# Singularities of orbit closures in module varieties (part 1)

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# Representations of quivers

- Throughout the talks $\Bbbk$ will denote an allgebraically closed field (of arbitrary characteristic).
- Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver, where $Q_0$ is the set of vertices and $Q_1$ is the set of arrows:

$$s(\alpha) \xrightarrow{\alpha} t(\alpha).$$

- $\mathrm{Rep}(Q)$ is the category of representations of $Q$:

OBJECT:

$$V = \Big( (V_i \text{ - vector space over } \Bbbk\ )_{i \in Q_0}, \quad (V_\alpha \colon V_{s(\alpha)} \xrightarrow{\Bbbk\text{-linear}} V_{t(\alpha)})_{\alpha \in Q_1} \Big),$$

MORPHISM:

$$f = (\, f_i \colon V_i \xrightarrow{\Bbbk\text{-linear}} W_i)_{i \in Q_0} \colon V \to W,$$

such that the following diagram commutes for any $\alpha \in Q_1$:

$$\begin{array}{ccc} V_{s(\alpha)} & \xrightarrow{V_\alpha} & V_{t(\alpha)} \\ {\scriptstyle f_{s(\alpha)}}\downarrow & & \downarrow{\scriptstyle f_{t(\alpha)}} \\ W_{s(\alpha)} & \xrightarrow{W_\alpha} & W_{t(\alpha)}. \end{array}$$

- $\mathrm{rep}(Q)$ denotes the subcategory of finite dimensional representations in $\mathrm{Rep}(Q)$, and for them the dimension vector is defined:

$$\mathbf{dim}\, V = (\, \dim_\Bbbk V_i\, )_{i \in Q_0} \in \mathbb{N}^{Q_0}.$$

# Varieties of quiver representations

- Fix a dimension vector $\mathbf{d} = (d_i)_{i \in Q_0} \in \mathbb{N}^{Q_0}$.
- $\text{rep}_Q(\mathbf{d}) = \{ V \in \text{rep}(Q) \mid \forall_{i \in Q_0} \; V_i = \Bbbk^{d_i} \} = \prod_{\alpha \in Q_1} \mathbb{M}_{d_{t(\alpha)} \times d_{s(\alpha)}}(\Bbbk)$

  is a vector space (affine space, affine variety).
- Question: When two points $V, W \in \text{rep}_Q(\mathbf{d})$ are isomorphic as representations of $Q$ ?
- Answer: When they belong to the same orbit under the action of

$$\text{GL}(\mathbf{d}) = \prod_{i \in Q_0} \text{GL}_{d_i}(\Bbbk),$$

  via

$$(g_i)_{i \in Q_0} * (V_\alpha)_{\alpha \in Q_1} = (g_{t(\alpha)} \cdot V_\alpha \cdot g_{s(\alpha)}^{-1})_{\alpha \in Q_1}.$$

- If $M \in \text{rep}(Q)$, then $\mathcal{O}_M$ denotes the orbit in $\text{rep}_Q(\mathbf{dim}\, M)$ of points which are isomorphic to $M$ as representations of $Q$.
- We are mainly interested in orbit closure $\overline{\mathcal{O}}_M$, which is an affine variety, usually singular.
- Given two representations $M, N \in \text{rep}(Q)$ with $\mathbf{dim}\, M = \mathbf{dim}\, N$, we say that $M$ **degenerates** to $N$ (or $N$ is a **degeneration** of $M$) if $\mathcal{O}_N \subseteq \overline{\mathcal{O}}_M$. Then we write $M \leq_{deg} N$.
- $\leq_{deg}$ is a partial order on the set $\text{rep}(Q)/\simeq$ of isomorphism classes in $\text{rep}(Q)$.

## Example

Let $Q\colon 1 \xleftarrow{\alpha} 2$. If $M = (M_\alpha) \in \mathrm{rep}_Q(d_1, d_2) = \mathbb{M}_{d_1 \times d_2}(\Bbbk)$, then

$$\mathcal{O}_M = \{d_1 \times d_2 \text{ matrices of rank } = \mathrm{rk}(M_\alpha)\}$$

and

$$\overline{\mathcal{O}}_M = \{d_1 \times d_2 \text{ matrices of rank } \leq \mathrm{rk}(M_\alpha)\}.$$

The closure $\overline{\mathcal{O}}_M$ is a nonsingular variety iff $\mathrm{rk}(M_\alpha)$ equals 0 or $\min(d_1, d_2)$.

## Example

Let $Q : 1 \circlearrowleft \alpha$, so the indecomposable representations in $\mathrm{rep}(Q)$ corresponds to Jordan blocks.

It turns out that the singular locus

$$\mathrm{Sing}(\overline{\mathcal{O}}_M) = \overline{\mathcal{O}}_M \setminus \mathcal{O}_M.$$

Consequently, $\overline{\mathcal{O}}_M$ is nonsingular iff the orbit $\mathcal{O}_M$ is closed, iff the representation $M$ is semisimple, iff the matrix $M_\alpha$ is diagonalizable.

## Lemma

*If* $0 \to U \to M \to V \to 0$ *is a short exact sequence in* $\mathrm{rep}(Q)$*, then* $M$ *degenerates to* $U \oplus V$*.*

## Proof.

For simplicity, assume that $U \in \mathrm{rep}_Q(\mathbf{dim}\, U)$ and $V \in \mathrm{rep}_Q(\mathbf{dim}\, V)$.
There is $M' = (M'_\alpha)_{\alpha \in Q_1} \in \mathcal{O}_M$ such that

$$M'_\alpha = \begin{bmatrix} U_\alpha & Z_\alpha \\ 0 & V_\alpha \end{bmatrix} \qquad \text{for any } \alpha \in Q_1.$$

Consider the affine line in $\mathrm{rep}_Q(\mathbf{dim}\, M)$ consisting of points

$$M(t) = \left( \begin{bmatrix} U_\alpha & t \cdot Z_\alpha \\ 0 & V_\alpha \end{bmatrix} \right)_{\alpha \in Q_1}, \quad t \in \Bbbk.$$

The claim follows from the fact that $M(0) \simeq U \oplus V$ while $M(t) \simeq M$ for $t \neq 0$. □

## Corollary

- *Given a filtration of representations* $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_c = M$*, we have*

$$M \leq_{deg} \bigoplus_{i=1}^{c} M_i / M_{i-1}.$$

- *The orbit* $\mathcal{O}_M$ *is closed iff the representation* $M$ *is semisimple.*

- Let $\leq_{ext}$ be a partial order on the set $\mathrm{rep}(Q)/\simeq$ generated by $M \leq_{ext} U \oplus V$, whenever there is a short exact sequence $0 \to U \to M \to V \to 0$ in $\mathrm{rep}(Q)$.
- Obviously $M \leq_{ext} N$ implies $M \leq_{deg} N$.
- The converse implication does not hold in general, as $M <_{ext} N$ implies that $N$ is decomposable, while there exist proper degenerations to indecomposable representations.

## Example (Riedtmann, 1986)

Let $Q : 1 \xrightarrow{\alpha} 2 \circlearrowleft \beta$ and consider the following two indecomposable representations

$$M : k \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} k^2 \circlearrowleft \begin{bmatrix}0&0\\1&0\end{bmatrix} \quad \text{and} \quad N : k \xrightarrow{\begin{bmatrix}0\\1\end{bmatrix}} k^2 \circlearrowleft \begin{bmatrix}0&0\\1&0\end{bmatrix}.$$

Then $M <_{deg} N$, as $M(0) = N$ while $M(t) \simeq M$ for all $t \neq 0$, where

$$M(t) : k \xrightarrow{\begin{bmatrix}t\\1\end{bmatrix}} k^2 \circlearrowleft \begin{bmatrix}0&0\\1&0\end{bmatrix}.$$

The degenerations $M <_{deg} N$ also follows from existence of a short exact sequence

$$0 \to Z \xrightarrow{\begin{pmatrix}f\\g\end{pmatrix}} Z \oplus M \to N \to 0, \quad \text{where} \quad Z : 0 \longrightarrow k^2 \circlearrowleft \begin{bmatrix}0&0\\1&0\end{bmatrix}.$$

Indeed, $\mathrm{Coker}\begin{pmatrix}f\\g\end{pmatrix} \simeq N$, while $\mathrm{Coker}\begin{pmatrix}f+t\cdot 1_Z\\g\end{pmatrix} \simeq M$ for almost all $t \neq 0$.

# Bound quivers and algebras

- Let $(Q, I)$ be a bound quiver, i.e. $I$ is a two-sided ideal in the path algebra $\Bbbk Q$ of $Q$.
- We denote by $\mathrm{Rep}(Q, I)$ and $\mathrm{rep}(Q, I)$ the corresponding subcategories of $\mathrm{Rep}(Q)$ and $\mathrm{rep}(Q)$, respectively.
- On a geometric level, $\mathrm{rep}_{Q,I}(\mathbf{d})$ is a closed $\mathrm{GL}(\mathbf{d})$-subvariety of $\mathrm{rep}_Q(\mathbf{d})$.

## Example

Let $Q = \quad 1 \xleftarrow{\beta} 2 \xleftarrow{\gamma} 3$, $I = (\beta\gamma)$ and $\mathbf{d} = (d_1, d_2, d_3)$. Then

$$\mathrm{rep}_{Q,I}(\mathbf{d}) = \{(V_\beta, V_\gamma) \in \mathbb{M}_{d_1 \times d_2}(k) \times \mathbb{M}_{d_2 \times d_3}(k) |\ V_\beta \circ V_\gamma = 0\}.$$

- Given an associative $\Bbbk$-algebra $A$ and $d \in \mathbb{N}$, one defines

$$\begin{aligned} \mathrm{mod}_A(d) &= \{A\text{-module structures on } \Bbbk^d\} \\ &= \{M : A \to \mathrm{End}_{\Bbbk}(\Bbbk^d)\text{-algebra homomorphism}\} \\ &= \{M : A \to \mathbb{M}_d(\Bbbk)\text{-algebra homomorphism}\}. \end{aligned}$$

- The group $\mathrm{GL}_d(\Bbbk)$ acts on $\mathrm{mod}_A(d)$ via

$$(g * M)(a) = g \cdot M(a) \cdot g^{-1} \qquad \forall\, a \in A.$$

- If $A$ is finitely generated ($A = \Bbbk\langle X_1, \cdots, X_l\rangle/J$), then

$$\operatorname{mod}_A(d) \simeq_{\mathrm{GL}_d(\Bbbk)} \operatorname{rep}_{Q,J}((d)),$$

  where $Q$ is the quiver with one vertex and $l$ loops. Hence $\operatorname{mod}_A(d)$ is an affine $\mathrm{GL}_d(\Bbbk)$-variety.
- Assume the algebra $A$ is finite dimensional. By the famous Gabriel's theorem, there is a bound quiver $(Q, I)$ and an equivalence

$$\mathcal{F}\colon \operatorname{mod} A \to \operatorname{rep}(Q, I).$$

### Theorem (Bongartz, 1991)

*Let $\mathcal{F}$ be the above equivalence and $M, N \in \operatorname{mod} A$ with $\mathcal{O}_N \subseteq \overline{\mathcal{O}}_M$. Then $\mathcal{O}_{\mathcal{F}N} \subseteq \overline{\mathcal{O}}_{\mathcal{F}M}$ and*

$$\operatorname{Sing}(M, N) = \operatorname{Sing}(\mathcal{F}M, \mathcal{F}N).$$

*In other words, $\mathcal{F}$ preserves degeneration order and the types of corresponding singularities.*

We say that two pointed varieties $(\mathcal{X}, x_0)$ and $(\mathcal{Y}, y_0)$ are *smoothly equivalent* if there is $(\mathcal{Z}, z_0)$ together with two smooth morphisms

$$\begin{array}{ccc} & (\mathcal{Z}, z_0) & \\ \swarrow & & \searrow \\ (\mathcal{X}, x_0) & & (\mathcal{Y}, y_0) \end{array}$$

In particular, $\widehat{\mathcal{O}}_{x_0,\mathcal{X}}[\![X_1, \cdots, X_s]\!] \simeq \widehat{\mathcal{O}}_{z_0,\mathcal{Z}} \simeq \widehat{\mathcal{O}}_{y_0,\mathcal{Y}}[\![Y_1, \cdots, Y_t]\!]$.
This is an equivalence relation and the equivalence classes are denoted by $\mathrm{Sing}(\mathcal{X}, x_0)$ and called *types of singularities*.

## Lemma

*Assume* $\dim_{x_0} \mathcal{X} - \dim_{y_0} \mathcal{Y} = r \geq 0$. *Then* $\mathrm{Sing}(\mathcal{X}, x_0) = \mathrm{Sing}(\mathcal{Y}, y_0)$ *iff*

$$\widehat{\mathcal{O}}_{x_0,\mathcal{X}}[\![X_1, \cdots, X_s]\!] \simeq \widehat{\mathcal{O}}_{y_0,\mathcal{Y}}[\![Y_1, \cdots, Y_{s+r}]\!]$$

*for some* $s \geq 0$. *If* $char(\Bbbk) = 0$ *then we may assume* $s = 0$.

## Example

$$\mathrm{Sing}(\{(x, y) \in \Bbbk^2 \mid x^2 - y^2 = 0\}, (0, 0)) = \mathrm{Sing}(\{(x, y) \in \Bbbk^2 \mid x^2 - y^2 - y^3 = 0\}, (0, 0)).$$

If $\mathcal{O}_N \subseteq \overline{\mathcal{O}}_M$ for modules or representations, then we define

$$\mathrm{Sing}(M, N) := \mathrm{Sing}(\overline{\mathcal{O}}_M, n_0), \quad \text{where} \quad n_0 \in \mathcal{O}_N.$$

# Characterization of degenerations

## Theorem (Z., 2000)

*Let $M, N \in \mathrm{rep}_Q(\mathbf{d})$. TFAE:*

1. *$M$ degenerates to $N$ ($M \leq_{deg} N$)*
2. *$\exists$ a short exact sequence $0 \to Z \to Z \oplus M \to N \to 0$*
3. *$\exists$ a short exact sequence $0 \to N \to M \oplus Z' \to Z' \to 0$*
4. *$\exists$ a regular map $\mu : \Bbbk \to \overline{\mathcal{O}}_M$ such that $\mu(0) \in \mathcal{O}_N$ and $\mu(t) \in \mathcal{O}_M$ for almost all $t \in \Bbbk$.*

- In the above theorem, $\mathrm{rep}_Q(\mathbf{d})$ can be replaced by $\mathrm{rep}_{Q,I}(\mathbf{d})$, or by $\mathrm{mod}_A(d)$.
- The implications (2) $\implies$ (1) and (3) $\implies$ (1) follow from Riedtmann's work.
- For the reverse implications one uses the fact that any point of $\overline{\mathcal{O}}_M$ can be connected with orbit $\mathcal{O}_M$ via an irreducible curve. In the language of $A$-modules, this leads to an $A$-$\Bbbk[[t]]$-bimodule $W$ such that

$$W/(W \cdot t) \simeq_A N \qquad \text{and} \qquad W \otimes_{\Bbbk[[t]]} \Bbbk((t)) \simeq M \otimes_{\Bbbk} \Bbbk((t)).$$

Next, one shows that

$$W/(W \cdot t^{h+1}) \simeq_A W/(W \cdot t^h) \oplus M, \quad \text{for } h \text{ large enough.}$$

# Hom-order

- Let $[X, Y] := \dim_{\Bbbk} \mathrm{Hom}(X, Y)$ for any $A$-modules $X$ and $Y$.
- Assume that $\mathbf{dim}\, M = \mathbf{dim}\, N$. We write $M \leq_{hom} N$ if

$$[X, M] \leq [X, N] \qquad \text{and} \qquad [M, X] \leq [N, X]$$

  for any (indecomposable) module $X \in \mathrm{mod}\, A$.
- Observe that $M \leq_{deg} N$ implies $M \leq_{hom} N$, by applying functors $\mathrm{Hom}_A(X, -)$ and $\mathrm{Hom}_A(-, X)$ for a short exact sequence of the form $0 \to Z \to Z \oplus M \to N \to 0$.
- $\leq_{hom}$ is a partial order on $\mathrm{mod}\, A/\simeq$, and similar definitions can be made for quiver representations.

## Example

Let

$$Q : 1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2 \underset{\delta}{\overset{\gamma}{\leftleftarrows}} 3, \qquad I = \langle \alpha\delta, \beta\gamma, \alpha\gamma - \beta\delta \rangle,$$

and consider the following indecomposable representations:

$$M : k \underset{[0,1]}{\overset{[1,0]}{\leftleftarrows}} k^2 \underset{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\leftleftarrows}} k, \qquad U_\lambda : k \underset{[\lambda]}{\overset{[1]}{\leftleftarrows}} k \leftleftarrows 0, \qquad V_\mu : 0 \leftleftarrows k \underset{[1]}{\overset{[\mu]}{\leftleftarrows}} k.$$

Then $M \leq_{hom} U_\lambda \oplus V_\mu$ for any $\lambda, \mu \in \Bbbk$, while $M \leq_{deg} U_\lambda \oplus V_\mu$ only if $\lambda + \mu = 0$.

## Theorem (Bongartz 1996, 1995; Z. 1998)

*Let $Q$ be a Dynkin or an extended Dynkin quiver. Then*

$$\leq_{ext} \quad \equiv \quad \leq_{deg} \quad \equiv \quad \leq_{hom}$$

## Theorem (Z. 1999)

*Let $A$ be an algebra of finite representation type. Then*

$$\leq_{deg} \quad \equiv \quad \leq_{hom}$$

# Idea of the proof $M \leq_{hom} N \implies M \leq_{deg} N$ for repr. finite algebras

- Let $A$ be a representations finite algebra and $X$ be a direct sum of the indecomposable $A$-modules, so $E = \mathrm{End}_A(X)$ is the Auslander algebra of $A$.
- Existence of a short exact sequence of the form

$$0 \to Z \to Z \oplus M \to N \to 0$$

  for some module $Z$ is equivalent to existence of a right $E$-submodule $U \subseteq \mathrm{Hom}_A(X, N)$ with $\mathbf{dim}_E(U) = \mathbf{dim}_E \mathrm{Hom}_A(X, M)$.
- Hence the claim follows from the following fact:

**Lemma**

*Let $W$ be a right $E$-submodule of $\mathrm{Hom}_A(X, N)$ such that $\mathbf{dim}_E(W) \geq \mathbf{dim}_E \mathrm{Hom}_A(X, M)$. Then there is an $E$-submodule $U \subseteq W$ with $\mathbf{dim}_E(U) = \mathbf{dim}_E \mathrm{Hom}_A(X, M)$.*

- The above lemma can be proved by induction on $\mathbf{dim}_E(W)$, where the first step is obvious.
- The induction step follows from existence of Auslander-Reiten sequences.