On (Large) (Co)silting Bijections

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joint work in progress w/ Lidia Angeleri Hügel

Perspectives in Tilting Theory and Related Topics

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§1. Motivation

$K^b(\mathrm{proj}\,\Lambda)$

[Gar23, 1.1]

$K^{[-1,0]}(\mathrm{proj}\,\Lambda) =: K_\Lambda$

Λ: f.d. $\overline{\mathbb{K}}$-alg.

mod Λ

$$\begin{tikzcd}
 & & 2\text{-silt}(\Lambda) \arrow[d, leftrightarrow, "1:1"] \\
 & \text{inj-thick}(K_\Lambda) \arrow[dl, leftrightarrow, "1:1"', "\beta" near end, "i"' near start] \arrow[r, leftrightarrow, "1:1"] \arrow[ddd, leftrightarrow, "1:1", "W" near start, "\mathcal{T}"' near end] & \text{silt}(K_\Lambda) \\
c\text{-cot}(K_\Lambda) \arrow[rr, leftrightarrow, "(_^\vee, _^\wedge)"'] \arrow[d, leftrightarrow, "1:1", "H^0" near start, "(H^0)^{-1}"' near end] & & \text{silt}(K_\Lambda)/\sim_{add} \arrow[d, leftrightarrow, "1:1", "H^0"' near start, "(H^0)^{-1}" near end] \\
f\text{-tors}(\Lambda) \arrow[rr, leftrightarrow, "1:1", "P" near start, "gen"' near end] \arrow[dr, leftrightarrow, "1:1"', "\alpha" near start, "\text{filt}\circ\text{gen}" near end] & & s\tau\text{-tilt}(\Lambda) \\
 & \ell\text{-wide}(\Lambda) \arrow[ur, leftrightarrow, "1:1"'] &
\end{tikzcd}$$

$K^b(\mathrm{proj}\,\Lambda)$

[AIR14, 2.7 & 3.2]

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$$\begin{tikzcd}
 & & \text{2-silt}(\Lambda) \arrow[d, shift left, "H^0"] \arrow[d, phantom, "1:1"'] \\
f\text{-tors}(\Lambda) \arrow[rr, shift left, "gen"] \arrow[rr, phantom, "1:1"] & & s\tau\text{-tilt}(\Lambda) \arrow[ll, shift left, "P"] \arrow[u, shift left, "(H^0)^{-1}"]
\end{tikzcd}$$

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[MŠ17, 3.10]

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 & & \text{2-silt}(\Lambda) \arrow[d, shift left, "(H^0)^{-1}"] \arrow[d, shift right, phantom, "1:1"'] \\
f\text{-tors}(\Lambda) \arrow[rr, shift left, "\mathcal{P}"] \arrow[rr, phantom, "1:1"] \arrow[dr, shift left, "\alpha"] & & s\tau\text{-tilt}(\Lambda) \arrow[ll, shift left, "\mathrm{gen}"] \arrow[u, shift left, "H^0"] \arrow[dl, "1:1"] \\
 & l\text{-wide}(\Lambda) \arrow[ul, shift left, "\mathrm{filt}\circ\mathrm{gen}", "1:1"'] &
\end{tikzcd}
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[PZ23, 3.6]

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\text{c-cot}(K_\Lambda) \arrow[dd, leftrightarrow, "1:1", "H^0"' near start, "(H^0)^{-1}"' near end] & & \text{2-silt}(\Lambda) \arrow[dd, leftrightarrow, "1:1", "H^0" near start, "(H^0)^{-1}" near end] \\
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\text{c-cot}(K_\Lambda) \arrow[urr, leftrightarrow, "\beta \quad 1:1 \quad i"] \arrow[rrrr, leftrightarrow, "1:1"', "(-^\vee, -^\wedge)"] \arrow[ddd, leftrightarrow, "H^0", "1:1"', "(H^0)^{-1}"] & & {[AT22]} & & \mathrm{silt}(K_\Lambda)/\sim_{\mathrm{add}} \arrow[ddd, leftrightarrow, "H^0"', "1:1", "(H^0)^{-1}"'] \\
 & & & & \\
 & & & & \\
\text{f-tors}(\Lambda) \arrow[rrrr, leftrightarrow, "P \quad 1:1 \quad \mathrm{gen}"] \arrow[drr, leftrightarrow, "\alpha \quad 1:1"', "\mathrm{filt}\circ\mathrm{gen}"] & & & & s\tau\text{-tilt}(\Lambda) \\
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(1) $\mathcal{K}_\Lambda := \mathcal{K}^{[-1,0]}(\mathrm{proj}\,\Lambda) \subseteq \mathcal{K}^b(\mathrm{proj}\,\Lambda)$ is a reduced 0-Auslander extriangulated category:

- it is closed under extensions in $\mathcal{K}^b(\mathrm{proj}\,\Lambda)$ (thus, $\mathfrak{s}$-conflations are induced by distinguished triangles);

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- $\mathrm{Proj}_{\mathbb{E}}(\mathcal{K}_\Lambda) \cap \mathrm{Inj}_{\mathbb{E}}(\mathcal{K}_\Lambda) = \{0\}$;
- it is hereditary, since $\mathbb{E}^2(-,?) = \mathrm{Hom}_{\mathcal{K}^b(\mathrm{proj}\,\Lambda)}(-,?[2])|_{\mathcal{K}_\Lambda^{op} \times \mathcal{K}} = 0$;

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Thus, it has a nice theory of silting ($\Leftrightarrow$ tilting $\Leftrightarrow$ cotilting) mutation by [GNP23].

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(2) Moreover, for any ring A, the following are also red. 0-Auslander:

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(2) Moreover, for any ring A, the following are also red. 0-Auslander:

- *the category of large projective presentations* $\mathcal{K}^2_P := \mathcal{K}^{[-1,0]}(\mathrm{Proj}\,A) \subseteq \mathcal{K}^b(\mathrm{Proj}\,A)$;
- *the category of large injective copresentations* $\mathcal{K}^2_I := \mathcal{K}^{[0,1]}(\mathrm{Inj}\,A) \subseteq \mathcal{K}^b(\mathrm{Inj}\,A)$.

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2. Close the torsion pair under direct limits to obtain $(\vec{\mathcal{T}},\vec{\mathcal{F}}) \in \operatorname{tor}(\operatorname{Mod}\Lambda)$.

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3. Associate to it a "large cosilting module" $C \in \mathrm{Mod}\,\Lambda$ such that $\vec{\mathcal{F}} = \mathrm{Cogen}(C)$.
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5. Restrict the torsion pair $({}^{\perp_0}C', \operatorname{Cogen}(C')) \in \operatorname{tor}(\operatorname{Mod}\Lambda)$ to obtain the mutation $(\mathcal{T}', \mathcal{F}') := ({}^{\perp_0}C' \cap \operatorname{mod}\Lambda, \operatorname{Cogen}(C') \cap \operatorname{mod}\Lambda) \in \operatorname{tor}(\operatorname{mod}\Lambda)$ of $(\mathcal{T}, \mathcal{F}) \in \operatorname{tor}(\operatorname{mod}\Lambda)$.

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* However, important questions regarding cosilting mutation remain. We hope to find some answers in $\mathbb{R}^2_I$.

$$
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 & & 2\text{-silt}(\Lambda) \arrow[d, leftrightarrow, "1:1"] \\
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\text{c-cot}(K_\Lambda) \arrow[ur, leftrightarrow, "1:1", "\beta\ \iota"'] \arrow[urr, leftrightarrow, "(\_^\vee,\ \_^\wedge)"'] \arrow[d, leftrightarrow, "H^0", "(H^0)^{-1}"', "1:1"] & & \\
f\text{-tors}(\Lambda) \arrow[rr, leftrightarrow, "P", "gen"', "1:1"] & & s\tau\text{-tilt}(\Lambda) \\
 & l\text{-wide}(\Lambda) \arrow[ul, leftrightarrow, "\alpha", "\text{filt}\circ\text{gen}"', "1:1"] \arrow[ur, leftrightarrow, "1:1"] &
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<u>Qst.</u> Mod Λ?

$$
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c\text{-}\mathrm{cot}(\mathcal{K}_\Lambda) \arrow[rr, leftrightarrow, "(\_^{\vee},\,\_^{\wedge})"'] \arrow[d, leftrightarrow, "H^0"', "1:1"] \arrow[ur, "i", phantom] & & \mathrm{silt}(\mathcal{K}_\Lambda)/\sim_{add} \arrow[d, leftrightarrow, "H^0", "1:1"'] \\
f\text{-}\mathrm{tors}(\Lambda) \arrow[rr, leftrightarrow, "P", "1:1"'] \arrow[dr, leftrightarrow, "\alpha"', "1:1"] \arrow[u, "(H^0)^{-1}", phantom] & & s\tau\text{-}\mathrm{tilt}(\Lambda) \arrow[u, "(H^0)^{-1}"', phantom] \\
 & l\text{-}\mathrm{wide}(\Lambda) \arrow[ur, leftrightarrow, "1:1"'] \arrow[ul, "\mathrm{filt}\circ\mathrm{gen}", phantom] \arrow[uuu, "\sigma", phantom] &
\end{tikzcd}
$$

Qst. Mod Λ?

Qst. Cosilting?

§2. Large (co)silting theory

<u>Dfn.</u> [AMV15, 4.1] A (large) *silting complex* is a complex $\sigma \in \mathbb{K}^b(\mathrm{Proj}\,A)$ such that

(i) $\mathrm{Hom}_{\mathbb{K}^b(\mathrm{Proj}A)}(\sigma, \sigma^{(I)}[i]) = 0$ for all sets I and $i > 0$;

(ii) the smallest triangulated subcategory of $\mathbb{K}^b(\mathrm{Proj}\,A)$ which contains $\mathrm{Add}(\sigma)$ is $\mathbb{K}^b(\mathrm{Proj}\,A)$.

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Furthermore, we say it is *2-term* if it is concentrated in degrees -1 and 0, and we denote the class of such complexes by $2\text{-}\mathrm{Silt}(A)$.

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Dfn. [AMV15, 3.7 & 4.9] Let $T \in \mathrm{Mod}\,A$ and $\sigma \in \mathcal{K}^2_p$ be such that $H^0(\sigma) = T$. Then T is a *silting module* (w.r.t. σ) and $\mathrm{Gen}\,T$ is a *silting (torsion) class* if

- $\sigma \in 2\text{-}\mathrm{Silt}(A)$;
- equivalently, $(\mathrm{Gen}\,T, T^{\perp_0}) \in \mathrm{tor}(\mathrm{Mod}\,A)$.

<u>Rmk</u>. [AMV15, 3.7, 3.11 & 4.9] We have the pollowing bijections:

$K^b(\mathrm{Proj}\,A)$

A: ring

$\mathrm{Mod}\,A$

$$\begin{tikzcd}
 & & \text{2-Silt}(A) \\
 & & \overline{U} \arrow[d, mapsto, "H^0"', "1:1"] \\
 & (\mathrm{Gen}\,T, _) & T \arrow[l, mapsto] \\
 & \text{Silt-tors}(A) \arrow[r, leftrightarrow, "1:1"] & \mathrm{Silt}(A)/\sim_{\mathrm{Gen}}.
\end{tikzcd}$$

Stp. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be extriangulated and assume its subcategories are full, strict and additive.

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Thm. [AT22, 5.7] There exist mutually inverse bijections $\operatorname{bddhc-cot}(\mathcal{C}) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \operatorname{silt}(\mathcal{C})$ given by $\Phi(\mathcal{X}, \mathcal{Y}) := \mathcal{X} \cap \mathcal{Y}$ and $\Psi(\mathcal{M}) = (\mathcal{M}^{\vee}, \mathcal{M}^{\wedge})$.

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Rmk. For $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ reduced 0-Auslander, all cotorsion pairs are bounded and hereditary.

Dfn. An object $\sigma \in \mathcal{K}^2_p$ is *silting* if $\mathrm{Add}(\sigma) \subseteq \mathcal{K}^2_p$ is a silting subcategory; we denote their collection by $\mathrm{Silt}(\mathcal{K}^2_p)$.

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(2) By [AMV15, Lemma 3.6(3)], $(H^0)^{-1}(\mathrm{Gen}\,T) = \sigma^{\perp_1} \subseteq \mathbb{K}^2_p$.

Dfn. An object $\sigma \in \mathcal{K}^2_p$ is *silting* if $\mathrm{Add}(\sigma) \subseteq \mathcal{K}^2_p$ is a silting subcategory; we denote their collection by $\mathrm{Silt}(\mathcal{K}^2_p)$.

Prp. We can identify $2\text{-}\mathrm{Silt}(A)$ and $\mathrm{Silt}(\mathcal{K}^2_p)/\sim_{\mathrm{Add}}$.

Rmk. Let $\sigma \in \mathrm{Silt}(\mathcal{K}^2_p)$ and $T := H^0(\sigma)$.

(1) By [AT22], $(\mathrm{Add}(\sigma)^\vee, \mathrm{Add}(\sigma)^\wedge) \in (\mathrm{bddh})\,\text{c-cot}(\mathcal{K}^2_p)$.

(2) By [AMV15, Lemma 3.6(3)], $(H^0)^{-1}(\mathrm{Gen}\,T) = \sigma^{\perp_1} \subseteq \mathcal{K}^2_p$.

Prp. Let $\sigma \in \mathrm{Silt}(\mathcal{K}^2_p)$, $T := H^0(\sigma)$ and $\Sigma := (H^0)^{-1}(\mathrm{Gen}\,T)$. Then ${}^{\perp_1}\Sigma \cap \Sigma = \mathrm{Add}(\sigma)$ and $({}^{\perp_1}\Sigma, \Sigma) = (\mathrm{Add}(\sigma)^\vee, \mathrm{Add}(\sigma)^\wedge)$. Thus, there is an injection

$$\mathrm{Silt}(\mathcal{K}^2_p)/\sim_{\mathrm{Add}} \hookrightarrow \text{c-cot}(\mathcal{K}^2_p).$$

<u>Qst.</u> For which $(\mathcal{X},\mathcal{Y}) \in \text{c-cot}(\mathcal{K}_P^2)$ does there exist a $\sigma \in \mathcal{X} \cap \mathcal{Y}$ such that $\mathcal{X} \cap \mathcal{Y} = \operatorname{Add}(\bar{\sigma})$?

Qst. For which $(\mathcal{X}, \mathcal{Y}) \in$ c-cot$(\mathbb{K}^2_P)$ does there exist a $\sigma \in \mathcal{X} \cap \mathcal{Y}$ such that $\mathcal{X} \cap \mathcal{Y} = \mathrm{Add}(\sigma)$?

Lmm. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ have (arbitrary) coproducts, enough $\mathbb{E}$-projectives and $\mathbb{E}$-injectives, and let $(\mathcal{X}, \mathcal{Y}) \in$ c-cot$(\mathcal{C})$ be such that $\mathcal{X}$ is closed under coproducts and $\mathrm{pdim}(X) < \infty$ for all $X \in \mathcal{X}$. Then, for any $X' \in \mathcal{X}$, there exists a finite family $\{K_i\}_{i=0}^{n} \subseteq \mathcal{X} \cap \mathcal{Y}$ such that $X' \in \mathrm{add}(\bigoplus_{i=0}^{n} K_i)$.

Qst. For which $(\mathcal{X},\mathcal{Y}) \in$ c-cot$(\mathbb{K}^2_{\mathcal{P}})$ does there exist a $\sigma \in \mathcal{X}\cap\mathcal{Y}$ such that $\mathcal{X}\cap\mathcal{Y} = \mathrm{Add}(\sigma)$?

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Prp. Let $(\mathcal{X},\mathcal{Y}) \in$ c-cot$(\mathbb{K}^2_{\mathcal{P}})$ be such that $\mathcal{X}\cap\mathcal{Y}$ is closed under coproducts. Then there exists $\sigma \in \mathcal{X}\cap\mathcal{Y}$ such that $\mathcal{X}\cap\mathcal{Y} = \mathrm{Add}(\sigma)$

Proof sketch: 1. Apply the previous Lemma to $A \in \mathrm{Proj}_{\mathbb{E}}(\mathbb{K}^2_{\mathcal{P}}) \subseteq \mathcal{X}$ and define $\sigma := \bigoplus_{i=0}^n K_i$.

2. Show that $\mathrm{Add}(\sigma)$ is a silting subcategory of $\mathbb{K}^2_{\mathcal{P}}$.

3. Note that $\mathrm{Add}(\sigma) \subseteq \mathcal{X}\cap\mathcal{Y}$ and apply [AT22, Lemma 5.3]. ∎

Qst. For which $(\mathcal{X},\mathcal{Y}) \in$ c-cot$(\mathbb{K}^2_{\mathcal{P}})$ does there exist a $\sigma \in \mathcal{X}\cap\mathcal{Y}$ such that $\mathcal{X}\cap\mathcal{Y} = \mathrm{Add}(\sigma)$?

Lmm. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ have (arbitrary) coproducts, enough $\mathbb{E}$-projectives and $\mathbb{E}$-injectives, and let $(\mathcal{X},\mathcal{Y}) \in$ c-cot$(\mathcal{C})$ be such that $\mathcal{X}$ is closed under coproducts and $\mathrm{pdim}(X) < \infty$ for all $X \in \mathcal{X}$. Then, for any $X' \in \mathcal{X}$, there exists a finite family $\{K_i\}_{i=0}^{n} \subseteq \mathcal{X}\cap\mathcal{Y}$ such that $X' \in \mathrm{add}(\bigoplus_{i=0}^{n} K_i)$.

Prp. Let $(\mathcal{X},\mathcal{Y}) \in$ c-cot$(\mathbb{K}^2_{\mathcal{P}})$ be such that $\mathcal{X}\cap\mathcal{Y}$ is closed under coproducts. Then there exists $\sigma \in \mathcal{X}\cap\mathcal{Y}$ such that $\mathcal{X}\cap\mathcal{Y} = \mathrm{Add}(\sigma)$

Proof sketch: 1. Apply the previous Lemma to $A \in \mathrm{Proj}_{\mathbb{E}}(\mathbb{K}^2_{\mathcal{P}}) \subseteq \mathcal{X}$ and define $\sigma := \bigoplus_{i=0}^{n} K_i$.

2. Show that $\mathrm{Add}(\sigma)$ is a silting subcategory of $\mathbb{K}^2_{\mathcal{P}}$.

3. Note that $\mathrm{Add}(\sigma) \subseteq \mathcal{X}\cap\mathcal{Y}$ and apply [AT22, Lemma 5.3]. ∎

Dfn. A cotorsion pair $(\mathcal{X},\mathcal{Y})$ in $\mathbb{K}^2_{\mathcal{P}}$ is *silting* if it is complete and its kernel is closed under coproducts.

Cor. We have the following commutative diagram of bijections:

$$
\begin{tikzcd}
K^b(\mathrm{Proj}A) & & & & 2\text{-}\mathrm{Silt}(A) \arrow[d, leftrightarrow, "1:1"] \\
K_p^2 & \mathrm{Silt\text{-}cot}(K_p^2) \arrow[rrr, leftrightarrow, "1:1"'] \arrow[dddd, leftrightarrow, "1:1"'] & & & \mathrm{Silt}(K_p^2)/\sim_{\mathrm{Add}} \arrow[dddd, leftrightarrow, "1:1"] \\
& & (\mathrm{Add}(\overline{U})^{\vee}, \mathrm{Add}(\overline{U})^{\wedge}) \arrow[d, equal] & \overline{U} \arrow[l, mapsto] \arrow[dd, mapsto, "H^0"'] & \\
A: \text{ring} & & (\_, (H^0)^{-1}(\mathrm{Gen}T)) & & \\
& & (\mathrm{Gen}T, \_) \arrow[u, mapsto] & T \arrow[l, mapsto] & \\
\mathrm{Mod}A & \mathrm{Silt\text{-}tors}(A) \arrow[rrr, leftrightarrow, "1:1"] & & & \mathrm{Silt}(A)/\sim_{\mathrm{Gen}}.
\end{tikzcd}
$$

Rmk. [BP17, 3.5(2)] We have the following bijections:

A: ring

$$
\begin{tikzcd}
K^b(\mathrm{Inj}\,A) & & \omega \arrow[d, mapsto, "H^0"'] & \text{2-Cosilt}(A) \arrow[dd, leftrightarrow, "1:1"] \\
& (\_, \mathrm{Cogen}\, C) & C \arrow[l, mapsto] & \\
\mathrm{Mod}\,A & \text{Cosilt-torf}(A) \arrow[rr, leftrightarrow, "1:1"] & & \mathrm{Cosilt}(A)/\sim_{\mathrm{cogen.}}
\end{tikzcd}
$$

Rmk. By dual definitions and results, we obtain the commutative diagram of bijections

$$
\begin{tikzcd}[column sep=small]
K^b(\mathrm{Inj}\,A) & & & 2\text{-}\mathrm{Cosilt}(A) \arrow[d, leftrightarrow, "1:1"] \\
K_I^2 & \mathrm{Cosilt}\text{-}\mathrm{cot}(K_I^2) \arrow[rr, leftrightarrow, "1:1"'] \arrow[ddd, leftrightarrow, "1:1"'] & & \mathrm{Cosilt}(K_I^2)/\sim_{\mathrm{Prod}} \arrow[ddd, leftrightarrow, "1:1"] \\
& & (\mathrm{Prod}(\omega)^{\vee}, \mathrm{Prod}(\omega)^{\wedge}) \arrow[d, phantom, "\shortparallel"] & \omega \arrow[l, mapsto] \arrow[dd, mapsto, "H^0"'] \\
A:\text{ring} & & ((H^0)^{-1}(\mathrm{Cogen}\,C), \_\_) & \\
& & (\_\_, \mathrm{Cogen}\,C) \arrow[u, mapsto] & C \arrow[l, mapsto] \\
\mathrm{Mod}\,A & \mathrm{Cosilt}\text{-}\mathrm{torf}(A) \arrow[rr, leftrightarrow, "1:1"] & & \mathrm{Cosilt}(A)/\sim_{\mathrm{Cogen.}}
\end{tikzcd}
$$

§3. Inverting Ingalls-Thomas' maps

<u>Rmk.</u> In an extriangulated category $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$, the common generalization of kernel-cokernel pairs and distinguished triangles are called *$\mathfrak{s}$-conflations*, and they are denoted by

$$A \longrightarrow B \longrightarrow C \dashrightarrow .$$

Rmk. In an extriangulated category $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$, the common generalization of kernel-cokernel pairs and distinguished triangles are called $\mathfrak{s}$-*conflations*, and they are denoted by

$$A \xrightarrow{\text{inflation}} B \longrightarrow \underset{\text{cone}}{C} \dashrightarrow .$$

Dfn. Let $\mathcal{R}, \mathcal{S} \subseteq \mathscr{C}$.

(a) $\mathrm{Cone}(\mathcal{R}, \mathcal{S})$ denotes the class of cones of inflations from objects in $\mathcal{R}$ to objects in $\mathcal{S}$.

Rmk. In an extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, the common generalization of kernel-cokernel pairs and distinguished triangles are called *$\mathfrak{s}$-conflations*, and they are denoted by

$$\underset{\text{cocone}}{A} \xrightarrow{\text{inflation}} B \xrightarrow{\text{deflation}} \underset{\text{cone}}{C} \dashrightarrow .$$

Dfn. Let $\mathcal{R}, \mathcal{S} \subseteq \mathcal{C}$.

(a) $\mathrm{Cone}(\mathcal{R}, \mathcal{S})$ denotes the class of cones of inflations from objects in $\mathcal{R}$ to objects in $\mathcal{S}$.

(b) $\mathrm{Cocone}(\mathcal{R}, \mathcal{S})$ denotes the class of cocones of deflations from objects in $\mathcal{R}$ to objects in $\mathcal{S}$.

Rmk. In an extriangulated category $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$, the common generalization of kernel-cokernel pairs and distinguished triangles are called $\mathfrak{s}$-*conflations*, and they are denoted by

$$\begin{tikzcd} A \arrow[r, "\text{inflation}"] & B \arrow[r, "\text{deflation}"] & C \arrow[r, dashed] & {} \\ \text{cocone} & \text{extension} & \text{cone} & \end{tikzcd}$$

Dfn. Let $\mathcal{R}, \mathcal{S} \subseteq \mathscr{C}$.

(a) $\mathrm{Cone}(\mathcal{R}, \mathcal{S})$ denotes the class of cones of inflations from objects in $\mathcal{R}$ to objects in $\mathcal{S}$.

(b) $\mathrm{Cocone}(\mathcal{R}, \mathcal{S})$ denotes the class of cocones of deflations from objects in $\mathcal{R}$ to objects in $\mathcal{S}$.

(c) $\mathcal{R} * \mathcal{S}$ denotes the class of extensions of objects in $\mathcal{S}$ by objects in $\mathcal{R}$.

Stp. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be extriangulated and assume its subcategories are full, strict and additive.

Dfn. For $\mathcal{S} \subseteq \mathscr{C}$, let $\beta(\mathcal{S}) := \{S \in \mathcal{S} \mid \exists\, S \to S' \to S'' \dashrightarrow,\ S' \in \mathcal{S} \Rightarrow S'' \in \mathcal{S}\}$ and $i(\mathcal{S}) := \{C \in \mathscr{C} \mid \exists\, C \to S \to C' \dashrightarrow$ with $S \in \mathcal{S}\}$.

Rmk. Let $\mathcal{S} \subseteq \mathscr{C}$.

(1) $\beta(\mathcal{S})$ is the biggest subcategory contained in $\mathcal{S}$ and closed under cones.

(2) $i(\mathcal{S})$ is the smallest subcategory containing $\mathcal{S}$ and closed under inflations.

Stp. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be extriangulated and assume its subcategories are full, strict and additive.

Dfn. For $\mathcal{S} \subseteq \mathcal{C}$, let $\beta(\mathcal{S}) := \{ S \in \mathcal{S} \mid \exists\, S \to S' \to S'' \dashrightarrow,\ S' \in \mathcal{S} \Rightarrow S'' \in \mathcal{S} \}$ and $i(\mathcal{S}) := \{ C \in \mathcal{C} \mid \exists\, C \to S \to C' \dashrightarrow$ with $S \in \mathcal{S} \}$.

Rmk. Let $\mathcal{S} \subseteq \mathcal{C}$.

(1) $\beta(\mathcal{S})$ is the biggest subcategory contained in $\mathcal{S}$ and closed under cones.

(2) $i(\mathcal{S})$ is the smallest subcategory containing $\mathcal{S}$ and closed under inflations.

(3) If $i(\mathcal{S}) \subseteq \mathcal{S}$ (i.e., $\mathcal{S}$ is closed under inflations), then $i(\beta(\mathcal{S})) \subseteq \mathcal{S}$.

Stp. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be extriangulated and assume its subcategories are full, strict and additive.

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(3) If $i(\mathcal{S}) \subseteq \mathcal{S}$ (i.e., $\mathcal{S}$ is closed under inflations), then $i(\beta(\mathcal{S})) \subseteq \mathcal{S}$.

Dfn. We say $\mathcal{S} \subseteq \mathscr{C}$ is

(a) *torsionfree* if $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$ and $i(\mathcal{S}) \subseteq \mathcal{S}$;

Stp. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be extriangulated and assume its subcategories are full, strict and additive.

Dfn. For $\mathcal{S} \subseteq \mathscr{C}$, let $\beta(\mathcal{S}) := \{ S \in \mathcal{S} \mid \forall\, S \to S' \to S'' \dashrightarrow,\ S' \in \mathcal{S} \Rightarrow S'' \in \mathcal{S} \}$ and $i(\mathcal{S}) := \{ C \in \mathscr{C} \mid \exists\, C \to S \to C' \dashrightarrow \text{ with } S \in \mathcal{S} \}$.

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Dfn. We say $\mathcal{S} \subseteq \mathscr{C}$ is

(a) *torsionfree* if $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$ and $i(\mathcal{S}) \subseteq \mathcal{S}$;

(b) *thick* if $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$, $\mathrm{smd}(\mathcal{S}) \subseteq \mathcal{S}$, $\mathrm{Cone}(\mathcal{S}, \mathcal{S}) \subseteq \mathcal{S}$ and $\mathcal{S} \supseteq \mathrm{Cocone}(\mathcal{S}, \mathcal{S})$;

Stp. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be extriangulated and assume its subcategories are full, strict and additive.

Dfn. For $\mathcal{S} \subseteq \mathcal{C}$, let $\beta(\mathcal{S}) := \{ S \in \mathcal{S} \mid \forall\, S \to S' \to S'' \dashrightarrow,\ S' \in \mathcal{S} \Rightarrow S'' \in \mathcal{S} \}$ and $i(\mathcal{S}) := \{ C \in \mathcal{C} \mid \exists\, C \to S \to C' \dashrightarrow$ with $S \in \mathcal{S} \}$.

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(2) $i(\mathcal{S})$ is the smallest subcategory containing $\mathcal{S}$ and closed under inflations.
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Dfn. We say $\mathcal{S} \subseteq \mathcal{C}$ is

(a) *torsionfree* if $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$ and $i(\mathcal{S}) \subseteq \mathcal{S}$;
(b) *thick* if $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$, $\mathrm{smd}(\mathcal{S}) \subseteq \mathcal{S}$, $\mathrm{Cone}(\mathcal{S}, \mathcal{S}) \subseteq \mathcal{S}$ and $\mathcal{S} \supseteq \mathrm{Cocone}(\mathcal{S}, \mathcal{S})$;
(c) *resolving* if $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$, $\mathrm{smd}(\mathcal{S}) \subseteq \mathcal{S}$, $\mathcal{S} \supseteq \mathrm{Cocone}(\mathcal{S}, \mathcal{S})$ and $\mathrm{Cone}(\mathcal{C}, \mathcal{S}) \supseteq \mathcal{C}$.

<u>Stp</u>. Suppose $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is hereditary.

<u>Prp</u>. Let $\mathcal{S} \subseteq \mathcal{C}$.

(a) If $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$, then $i(\mathcal{S})$ is the smallest torsionfree class containing $\mathcal{S}$.

<u>Stp</u>. Suppose $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is hereditary.

<u>Prp</u>. Let $\mathcal{S} \subseteq \mathcal{C}$.

(a) If $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$, then $i(\mathcal{S})$ is the smallest torsionfree class containing $\mathcal{S}$.

(b) If $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$, $\operatorname{smd}(\mathcal{S}) \subseteq \mathcal{S}$ and $\mathcal{S} \subseteq \operatorname{Cocone}(\mathcal{S}, \mathcal{S})$, then $\mathcal{B}(\mathcal{S})$ is the biggest thick subcategory contained in $\mathcal{S}$

<u>Stp</u>. Suppose $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is hereditary.

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(c) $\mathcal{S} \in \operatorname{thick}(\mathscr{C})$ if, and only if, $\mathcal{B}(i(\mathcal{S})) = \mathcal{S}$.

Stp. Suppose $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is hereditary.

Prp. Let $\mathcal{S} \subseteq \mathscr{C}$.

(a) If $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$, then $i(\mathcal{S})$ is the smallest torsionfree class containing $\mathcal{S}$.

(b) If $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$, $\mathrm{smd}(\mathcal{S}) \subseteq \mathcal{S}$ and $\mathcal{S} \subseteq \mathrm{Cocone}(\mathcal{S}, \mathcal{S})$, then $\beta(\mathcal{S})$ is the biggest thick subcategory contained in $\mathcal{S}$.

(c) $\mathcal{S} \in \mathrm{thick}(\mathscr{C})$ if, and only if, $\beta(i(\mathcal{S})) = \mathcal{S}$.

Dfn. $\mathcal{S} \in \mathrm{torf}(\mathscr{C})$ is *thickly generated* if $\exists\, \mathcal{S}' \in \mathrm{thick}(\mathscr{C})$ such that $\mathcal{S} = i(\mathcal{S}')$.

Stp. Suppose $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is hereditary.

Prp. Let $\mathcal{S} \subseteq \mathscr{C}$.

(a) If $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$, then $i(\mathcal{S})$ is the smallest torsionfree class containing $\mathcal{S}$.

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Dfn. $\mathcal{S} \in \mathrm{torf}(\mathscr{C})$ is *thickly generated* if $\exists\, \mathcal{S}' \in \mathrm{thick}(\mathscr{C})$ such that $\mathcal{S} = i(\mathcal{S}')$.

Cor. Let $\mathcal{S} \in \mathrm{torf}(\mathscr{C})$. Then $\mathcal{S}$ is thickly generated if and only if, $i(\beta(\mathcal{S})) = \mathcal{S}$. In particular, the maps $\mathrm{thick}(\mathscr{C}) \underset{\beta}{\overset{i}{\rightleftarrows}} \{\mathcal{S} \in \mathrm{torf}(\mathscr{C}) \mid \mathcal{S} \text{ is thickly generated}\}$ are mutually inverse.

Stp. Suppose $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is hereditary.

Prp. Let $\mathcal{S} \subseteq \mathscr{C}$.

(a) If $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$, then $i(\mathcal{S})$ is the smallest torsionfree class containing $\mathcal{S}$.

(b) If $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$, $\mathrm{smd}(\mathcal{S}) \subseteq \mathcal{S}$ and $\mathcal{S} \subseteq \mathrm{Cocone}(\mathcal{S}, \mathcal{S})$, then $\beta(\mathcal{S})$ is the biggest thick subcategory contained in $\mathcal{S}$.

(c) $\mathcal{S} \in \mathrm{thick}(\mathscr{C})$ if, and only if, $\beta(i(\mathcal{S})) = \mathcal{S}$.

Dfn. $\mathcal{S} \in \mathrm{torf}(\mathscr{C})$ is *thickly generated* if $\exists\, \mathcal{S}' \in \mathrm{thick}(\mathscr{C})$ such that $\mathcal{S} = i(\mathcal{S}')$.

Cor. Let $\mathcal{S} \in \mathrm{torf}(\mathscr{C})$. Then $\mathcal{S}$ is thickly generated if and only if, $i(\beta(\mathcal{S})) = \mathcal{S}$. In particular, the maps $\mathrm{thick}(\mathscr{C}) \underset{\beta}{\overset{i}{\rightleftarrows}} \{\mathcal{S} \in \mathrm{torf}(\mathscr{C}) \mid \mathcal{S} \text{ is thickly generated}\}$ are mutually inverse.

Rmk. If $(\mathcal{X}, \mathcal{Y}) \in \mathrm{cot}(\mathscr{C})$, then $\mathcal{X}$ is torsionfree.

__Stp__. Suppose $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is reduced 0-Auslander.

__Dfn__. Let $\mathcal{S} \subseteq \mathscr{C}$ be such that $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$. Then $\mathcal{S}$ *has enough injectives* if the induced extriangulated category $(\mathcal{S}, \mathbb{E}|_{\mathcal{S}}, \mathfrak{s}|_{\mathcal{S}})$ has enough ($\mathbb{E}|_{\mathcal{S}}$-) injectives (see [NP19]).

Stp. Suppose $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is reduced 0-Auslander.

Dfn. Let $\mathcal{S} \subseteq \mathscr{C}$ be such that $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$. Then $\mathcal{S}$ *has enough injectives* if the induced extriangulated category $(\mathcal{S}, \mathbb{E}|_{\mathcal{S}}, \mathfrak{s}|_{\mathcal{S}})$ has enough $(\mathbb{E}|_{\mathcal{S}}\text{-})$ injectives (see [NP19]).

Prp. [SaK] The maps $\{\mathcal{S} \in \mathrm{torf}(\mathscr{C}) \mid \mathcal{S} \text{ has enough injectives}\} \underset{G_1}{\overset{F_1}{\rightleftarrows}} \text{c-cot}(\mathscr{C})$ given by $F_1(\mathcal{S}) := (\mathcal{S}, \mathcal{S}^{\perp_1})$ and $G_1(\mathcal{X}, \mathcal{Y}) := \mathcal{X}$ are mutually inverse.

Stp. Suppose $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is reduced 0-Auslander.

Dfn. Let $\mathcal{S} \subseteq \mathscr{C}$ be such that $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$. Then $\mathcal{S}$ *has enough injectives* if the induced extriangulated category $(\mathcal{S}, \mathbb{E}|_{\mathcal{S}}, \mathfrak{s}|_{\mathcal{S}})$ has enough $(\mathbb{E}|_{\mathcal{S}}\text{-})$ injectives (see [NP19]).

Prp. [SaK] The maps $\{\mathcal{S} \in \operatorname{torp}(\mathscr{C}) \mid \mathcal{S} \text{ has enough injectives}\} \underset{G_1}{\overset{F_1}{\rightleftarrows}} \text{c-cot}(\mathscr{C})$ given by $F_1(\mathcal{S}) := (\mathcal{S}, \mathcal{S}^{\perp_1})$ and $G_1(\mathcal{X}, \mathcal{Y}) := \mathcal{X}$ are mutually inverse.

Qst. Does the previous bijection restrict to subcategories with enough injectives?

Stp. Suppose $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is reduced 0-Auslander.

Dfn. Let $\mathcal{S} \subseteq \mathscr{C}$ be such that $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$. Then $\mathcal{S}$ *has enough injectives* if the induced extriangulated category $(\mathcal{S}, \mathbb{E}|_{\mathcal{S}}, \mathfrak{s}|_{\mathcal{S}})$ has enough ($\mathbb{E}|_{\mathcal{S}}$-) injectives (see [NP19]).

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Qst. Does the previous bijection restrict to subcategories with enough injectives?

Rmk. [Gar23, 3.10] A thick subcategory $\mathcal{R}$ of a hereditary extriangulated category $\mathscr{C}$ has enough injectives if, and only if, $\mathcal{R} = \operatorname{thick}(\mathcal{U})$ for some presilting subcategory $\mathcal{U} \subseteq \mathscr{C}$ such that $\operatorname{Cone}(\mathcal{U}, \mathcal{U}) \subseteq \mathcal{U}$.

Prp. Let $(\mathcal{X}, \mathcal{Y}) \in \mathrm{Cosilt\text{-}cot}(\mathbb{K}^{[0,1]}(\mathrm{Inj}\,\Lambda))$ for Λ a f.d. $\overline{\mathbb{k}}$-alg. Then there exist $\mathbb{s}$-conflations

$$\omega_1 \to \omega_0 \xrightarrow{\psi} D(\Lambda) \dashrightarrow \quad \text{and} \quad D(\Lambda)[-1] \xrightarrow{\phi} \omega_1 \to \omega_0 \dashrightarrow$$

such that

(a) $\mathrm{Prod}(\omega_0 \oplus \omega_1) = \mathcal{X} \cap \mathcal{Y}$;

(b) ψ is a minimal right $\mathcal{X} \cap \mathcal{Y}$-approximation of $D(\Lambda)$ (+ dual statement).

Prp. Let $(\mathcal{X},\mathcal{Y}) \in \mathrm{Cosilt\text{-}cot}(\mathbb{K}^{[0,1]}(\mathrm{Inj}\,\Lambda))$ for Λ a f.d. $\mathbb{k}$-alg. Then there exist $\mathbb{s}$-conflations

$$\omega_1 \to \omega_0 \xrightarrow{\psi} D(\Lambda) \dashrightarrow \quad \text{and} \quad D(\Lambda)[-1] \xrightarrow{\phi} \omega_1 \to \omega_0 \dashrightarrow$$

such that

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Moreover, $i(\beta(\mathcal{X})) = \mathcal{X}$. In particular, $\mathcal{X}$ is thickly generated.

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Qst. Is there an "intrinsic" characterization for the subcategories $S \subseteq \mathbb{K}^2_I$ such that $S = \beta(\mathcal{X})$ for $(\mathcal{X},\mathcal{Y}) \in \text{Cosilt}(\mathbb{K}^2_I)$?

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