

On (Large) (Co)silting Bijections

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joint work in progress w/ Lidia Angeleri Hügel

Perspectives in Tilting Theory and Related Topics

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§1 Motivation

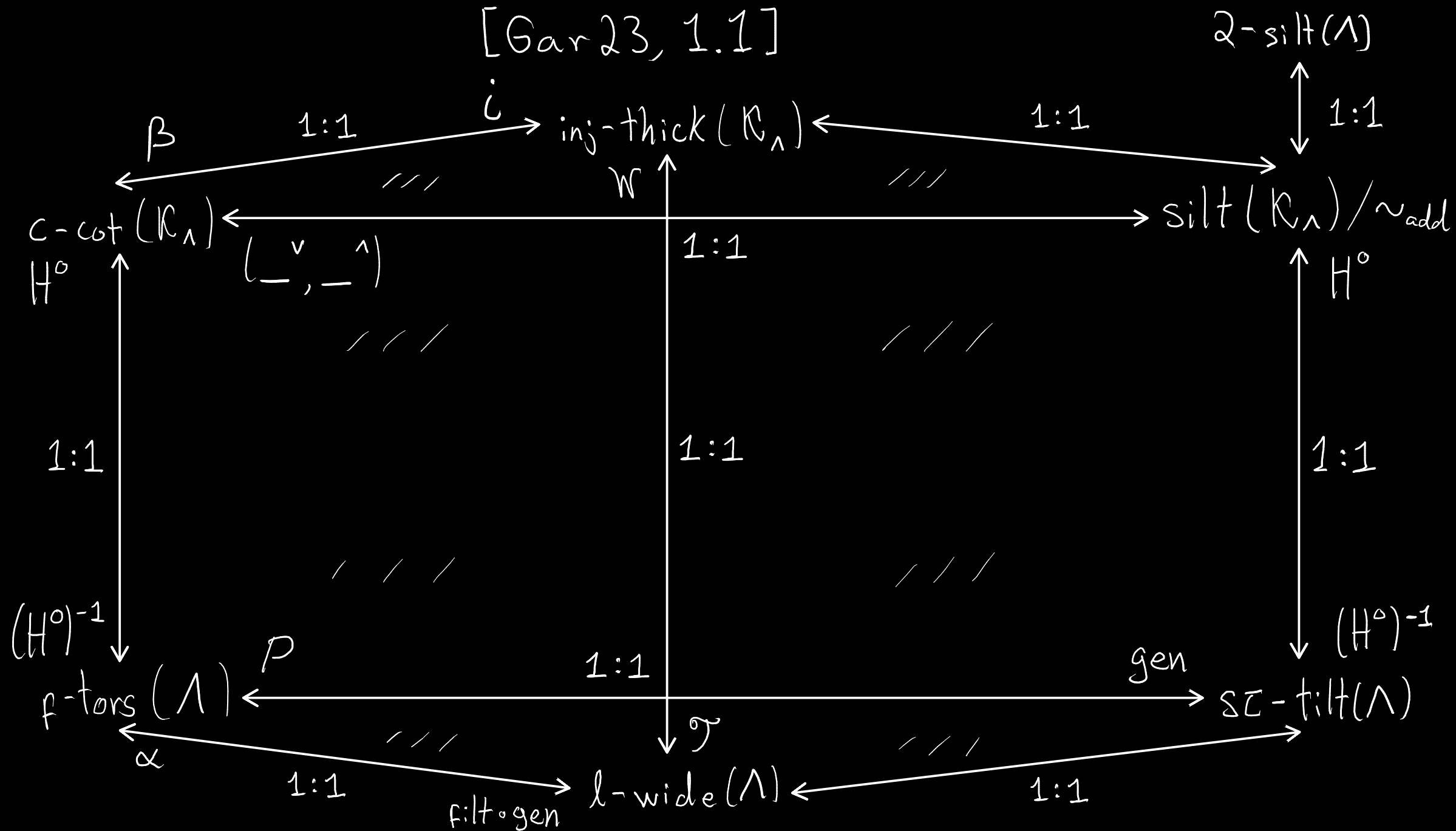
$$K^b(\text{proj } \Lambda)$$

$$K^{[-1,0]}(\text{proj } \Lambda) =: K_\Lambda$$

$$\Lambda: \text{p.d. } \overline{K}\text{-alg.}$$

$$\text{mod } \Lambda$$

[Gar23, 1.1]



$$K^b(\text{proj } \Lambda)$$

[AIR14, 2.7 & 3.2]

$$2\text{-silt}(\Lambda)$$

$$H^\circ$$

$$\Lambda: \text{p.d. } \overline{K}\text{-alg.}$$

$$1:1$$

$$(H^\circ)^{-1}$$

$$\text{mod } \Lambda$$

$$\text{f-tors}(\Lambda) \xleftarrow{P} \xrightarrow{1:1} \xrightarrow{\text{gen}} \text{st-tilt}(\Lambda)$$

$K^b(\text{proj } \Lambda)$

[MŠ17, 3.10]

$2\text{-silt}(\Lambda)$

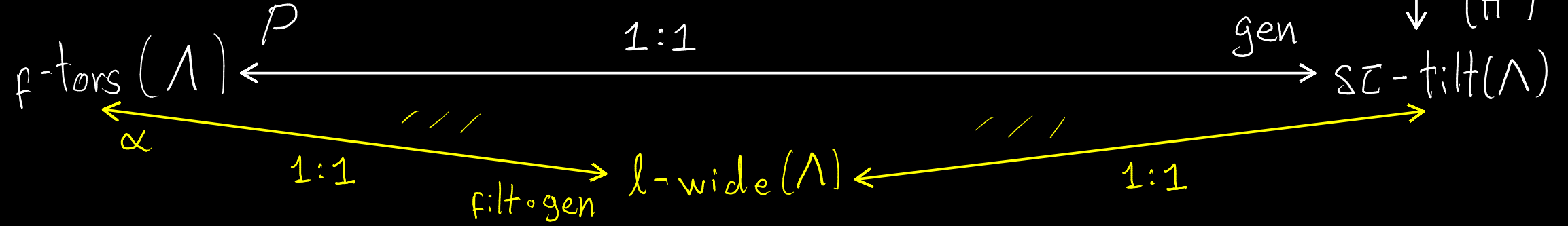
H°

$1:1$

$(H^\circ)^{-1}$

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[PZ23, 3.6]

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mod Λ

$$\begin{array}{c} \text{c-cot}(K_\Lambda) \\ \uparrow H^0 \end{array}$$

1:1

$$\downarrow (H^0)^{-1}$$

$$\text{f-tors}(\Lambda)$$

$\mathcal{P}$

1:1

gen

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α

1:1

filt $\circ$ gen

$$\text{l-wide}(\Lambda)$$

1:1

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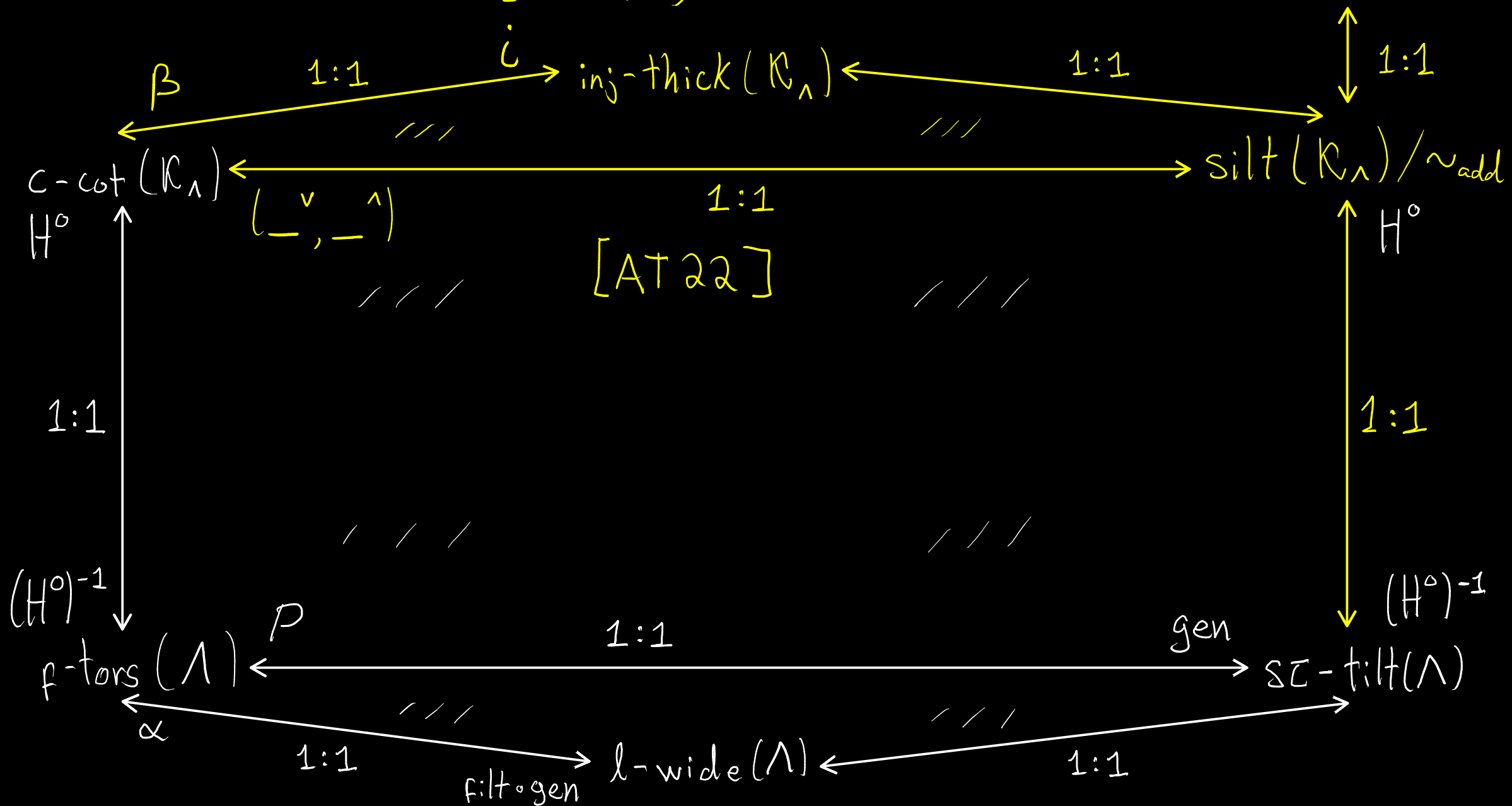
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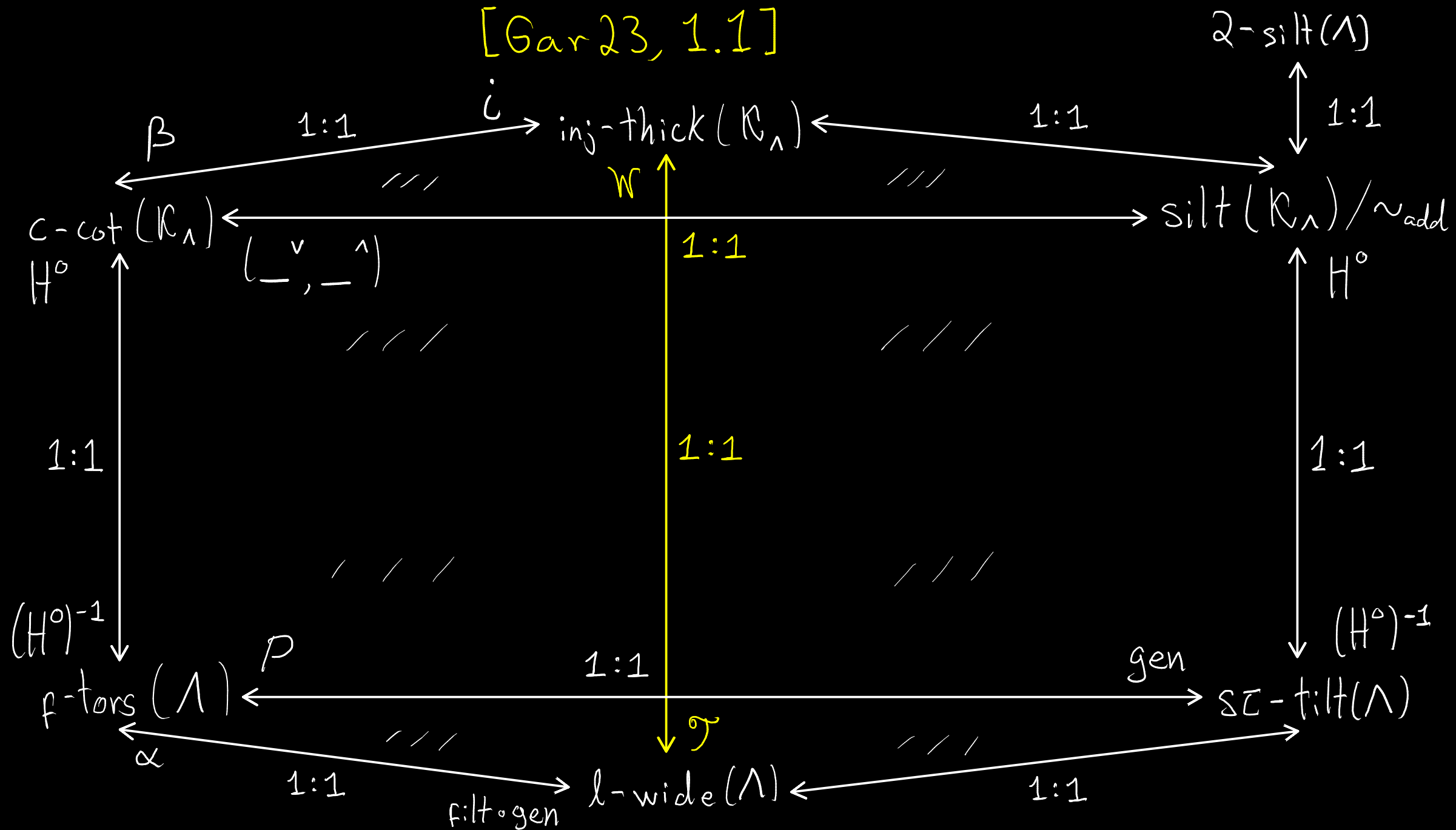
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(2) Moreover, for any ring A , the following are also red. 0-Auslander:

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(1) $\mathcal{K}_\Lambda := \mathcal{K}^{[-1,0]}(\text{proj } \Lambda) \subseteq \mathcal{K}^b(\text{proj } \Lambda)$ is a reduced O-Auslander extriangulated category:

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- the category of large injective copresentations $\mathcal{K}_I^2 := \mathcal{K}^{[0,1]}(\text{Inj } A) \subseteq \mathcal{K}^b(\text{Inj } A)$.

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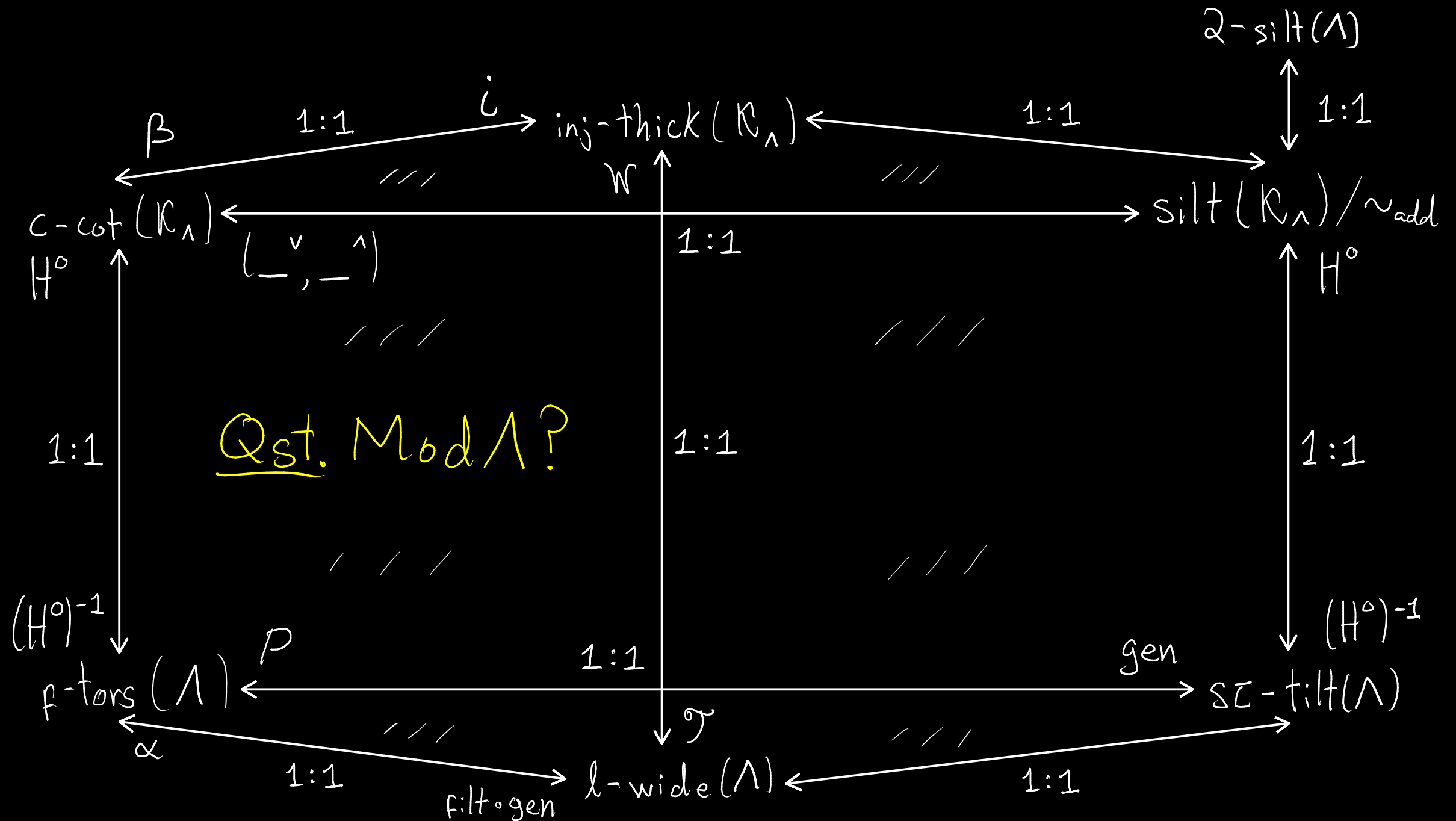
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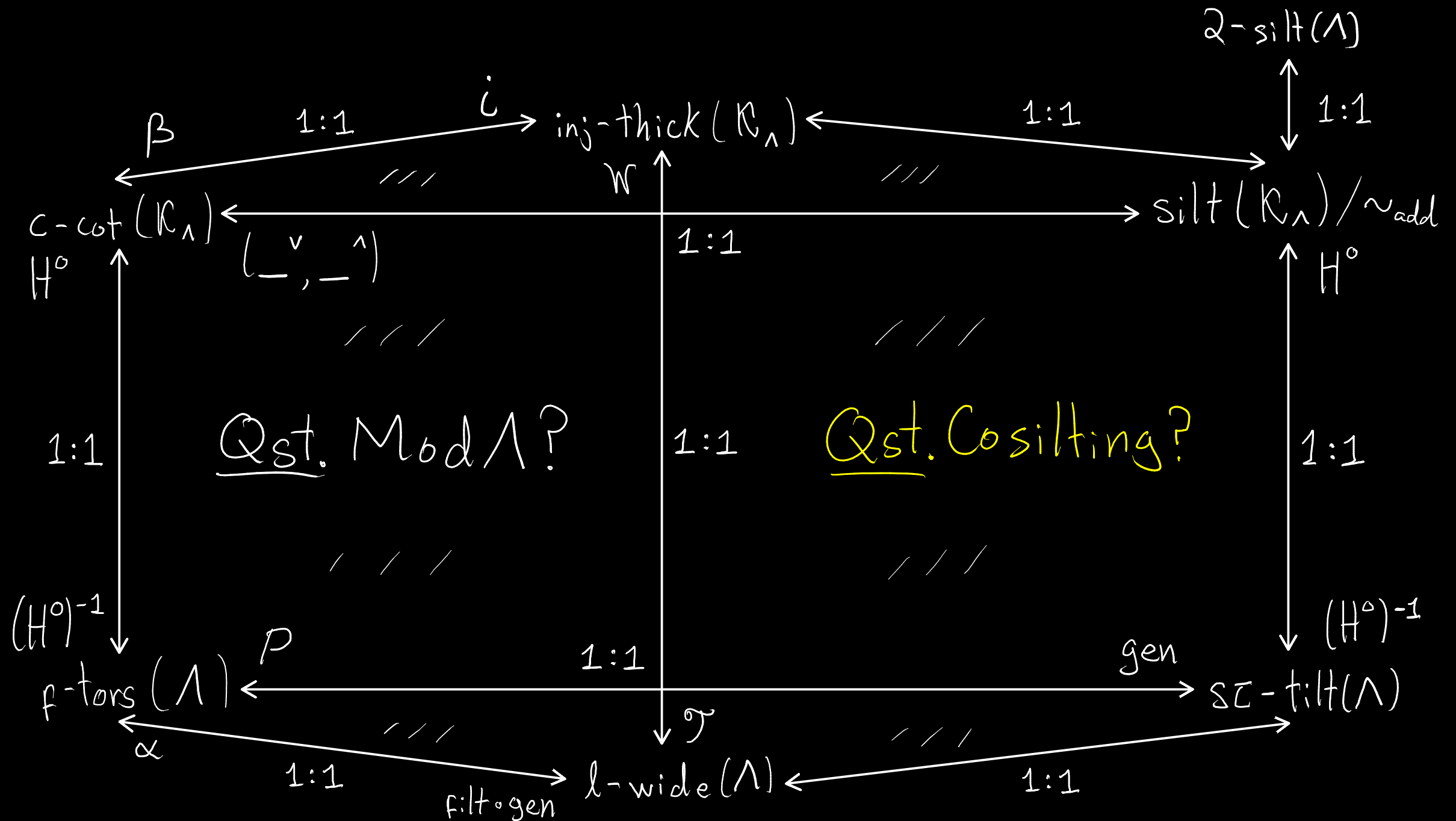
$(\mathcal{T}', \mathcal{F}') := ({}^{\perp_0} C' \cap \text{mod } \Lambda, \text{Cogen}(C') \cap \text{mod } \Lambda) \in \text{tor}(\text{mod } \Lambda)$ of $(\mathcal{T}, \mathcal{F}) \in \text{tor}(\text{mod } \Lambda)$.

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^{*} However, important questions regarding cosilting mutation remain. We hope to find some answers in $\mathbb{R}_{\text{I}}^2$.





§2. Large (co)silting theory

Defn. [AMV15, 4.1] A (large) silting complex is a complex $\sigma \in \mathcal{K}^b(\text{Proj } A)$ such that

- (i) $\text{Hom}_{\mathcal{K}^b(\text{Proj } A)}(\sigma, \sigma(I)[i]) = 0$ for all sets I and $i > 0$;
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Furthermore, we say it is *2-term* if it is concentrated in degrees -1 and 0 , and we denote the class of such complexes by $2\text{-Silt}(A)$.

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Defn. [AMV15, 3.7 & 4.9] Let $T \in \text{Mod } A$ and $\sigma \in \mathcal{K}_p^2$ be such that $H^0(\sigma) = T$. Then T is a *silting module* (w.r.t. σ) and $\text{Gen } T$ is a *silting (torsion) class* if

- $\sigma \in 2\text{-Silt}(A)$;

- equivalently, $(\text{Gen } T, T^{\perp_0}) \in \text{tor}(\text{Mod } A)$.

Rmk. [AMV15, 3.7, 3.11 & 4.9] We have the following bijections:

$$K^b(\text{Proj } A)$$

A : ring

$\text{Mod } A$

$$\mathcal{Q}\text{-Silt}(A)$$

$$\downarrow$$

$$1:1$$

$$H^\circ$$



$$(\text{Gen } T, _)$$



$$1:1$$

$$\text{Silt-tors}(A)$$



$$\text{Silt}(A)/\sim_{\text{Gen.}}$$

Stp. Let $(\mathcal{C}, E, s)$ be extriangulated and assume its subcategories are full, strict and additive.

Defn. $S \in \mathcal{C}$ is *silting* if $E^{>0}(S, S) = 0$ and $\text{thick}(S) = \mathcal{C}$.

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Defn. A complete cotorsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{C}$ is

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Thm. [AT22, 5.7] There exist mutually inverse bijections $\text{bddhc-cot}(\mathcal{C}) \xrightleftharpoons[\Psi]{\Phi} \text{silt}(\mathcal{C})$ given by $\Phi(\mathcal{X}, \mathcal{Y}) := \mathcal{X} \cap \mathcal{Y}$ and $\Psi(\mathcal{M}) = (\mathcal{M}^\vee, \mathcal{M}^\wedge)$.

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Rmk. For $(\mathcal{C}, \mathbb{E}, s)$ reduced 0-Auslander, all cotorsion pairs are bounded and hereditary.

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Rmk. Let $\sigma \in \text{Silt}(\mathcal{K}_P^2)$ and $T := H^0(\sigma)$.

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Prop. Let $\sigma \in \text{Silt}(\mathcal{K}_p^2)$, $T := H^0(\sigma)$ and $\Sigma := (H^0)^{-1}(\text{Gen} T)$. Then ${}^{\perp_1}\Sigma \cap \Sigma = \text{Add}(\sigma)$ and $({}^{\perp_1}\Sigma, \Sigma) = (\text{Add}(\sigma)^\vee, \text{Add}(\sigma)^\wedge)$. Thus, there is an injection

$$\text{Silt}(\mathcal{K}_p^2) / \sim_{\text{Add}} \hookrightarrow c\text{-cot}(\mathcal{K}_p^2).$$

Qst. For which $(X, Y) \in \text{c-cot}(\mathbb{C}_P^2)$ does there exist a $\sigma \in X \cap Y$ such that $X \cap Y = \text{Add}(\sigma)$?

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Lmm. Let $(\mathcal{C}, \mathbb{E}, \mathcal{S})$ have (arbitrary) coproducts, enough $\mathbb{E}$ -projectives and $\mathbb{E}$ -injectives, and let $(\mathcal{X}, \mathcal{Y}) \in \text{c-cot}(\mathcal{C})$ be such that $\mathcal{X}$ is closed under coproducts and $\text{pdim}(X) < \infty$ for all $X \in \mathcal{X}$. Then, for any $X' \in \mathcal{X}$, there exists a finite family $\{K_i\}_{i=0}^n \subseteq \mathcal{X} \cap \mathcal{Y}$ such that $X' \in \text{add}(\bigoplus_{i=0}^n K_i)$.

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Prop. Let $(\mathcal{X}, \mathcal{Y}) \in \text{c-cot}(\mathcal{K}_P^2)$ be such that $\mathcal{X} \cap \mathcal{Y}$ is closed under coproducts. Then there exists $\sigma \in \mathcal{X} \cap \mathcal{Y}$ such that $\mathcal{X} \cap \mathcal{Y} = \text{Add}(\sigma)$.

Proof sketch: 1. Apply the previous Lemma to $A \in \text{Proj}_{\mathbb{E}}(\mathcal{K}_P^2) \subseteq \mathcal{X}$ and define $\sigma := \bigoplus_{i=0}^n K_i$.

2. Show that $\text{Add}(\sigma)$ is a silting subcategory of $\mathcal{K}_P^2$.

3. Note that $\text{Add}(\sigma) \subseteq \mathcal{X} \cap \mathcal{Y}$ and apply [AT22, Lemma 5.3]. □

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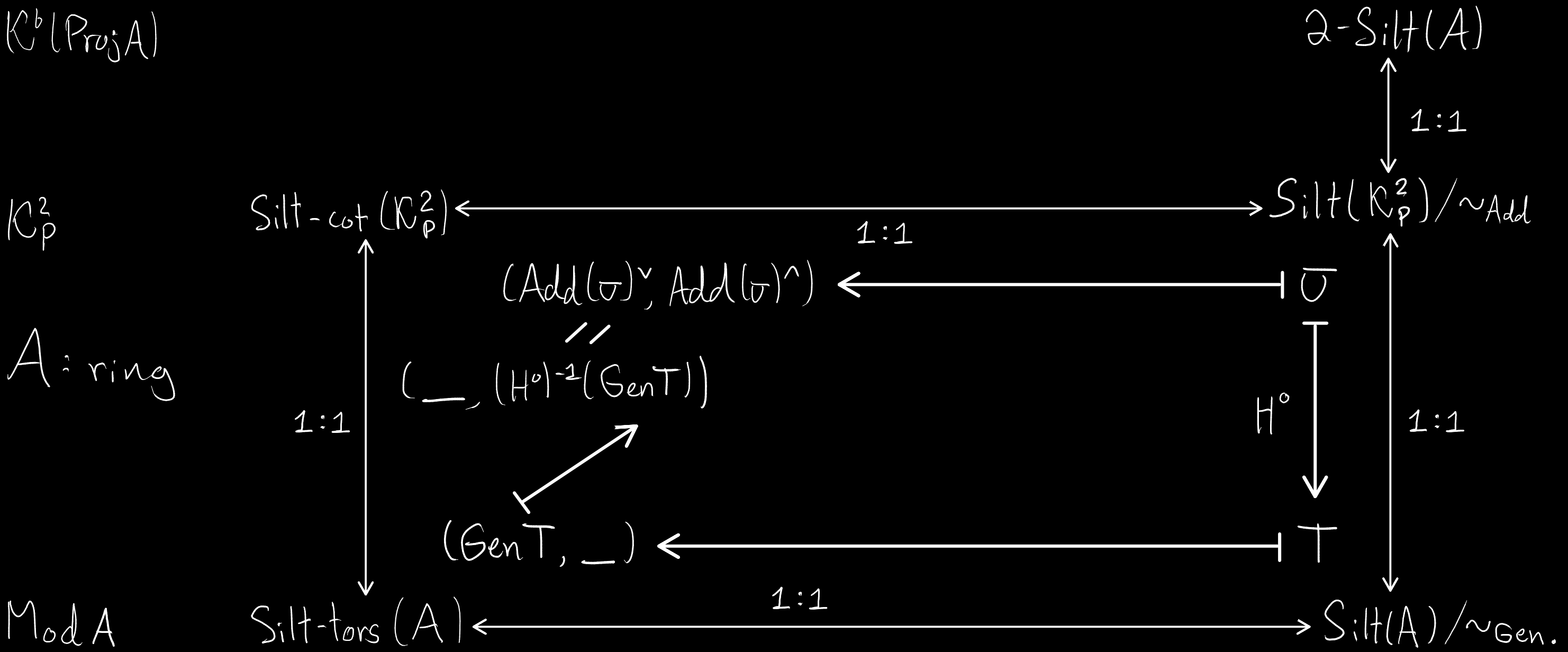
Proof sketch: 1. Apply the previous Lemma to $A \in \text{Proj}_{\mathbb{E}}(\mathcal{K}_P^2) \subseteq \mathcal{X}$ and define $\sigma := \bigoplus_{i=0}^n K_i$.

2. Show that $\text{Add}(\sigma)$ is a silting subcategory of $\mathcal{K}_P^2$.

3. Note that $\text{Add}(\sigma) \subseteq \mathcal{X} \cap \mathcal{Y}$ and apply [AT22, Lemma 5.3]. □

Def. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{K}_P^2$ is *silting* if it is complete and its kernel is closed under coproducts.

Cor. We have the following commutative diagram of bijections:



Rmk. [BP17, 3.5(2)] We have the following bijections:

$$K^b(\text{Inj } A)$$

A : ring

$\text{Mod } A$

$$\text{Cosilt-torf}(A)$$

$$(_, \text{Cogen } C)$$



1:1

C



ω

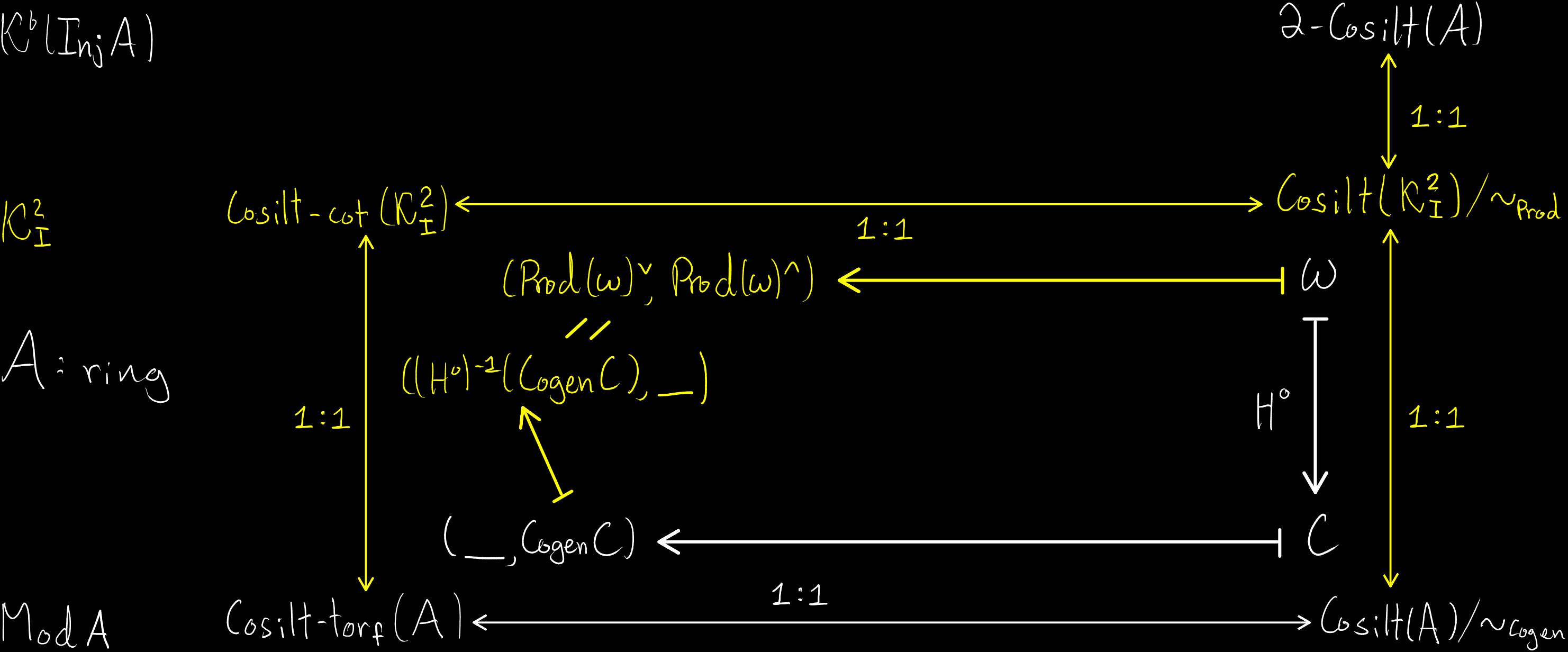
H^0

$$\mathfrak{Z}\text{-Cosilt}(A)$$

1:1

$$\text{Cosilt}(A)/\sim_{\text{Cogen.}}$$

Rmk. By dual definitions and results, we obtain the commutative diagram of bijections



§3. Inverting Ingalls-Thomas'
maps

Rmk. In an extriangulated category $(\mathcal{C}, \mathbb{E}, s)$, the common generalization of kernel-cokernel pairs and distinguished triangles are called s -conflations, and they are denoted by

$$A \rightarrow B \rightarrow C \dashrightarrow.$$

Rmk. In an extriangulated category $(\mathcal{C}, \mathbb{E}, s)$, the common generalization of kernel-cokernel pairs and distinguished triangles are called s -conflations, and they are denoted by

$$A \xrightarrow{\text{inflation}} B \longrightarrow C \dashrightarrow$$

cone

Def. Let $\mathcal{R}, \mathcal{S} \subseteq \mathcal{C}$.

(a) $\text{Cone}(\mathcal{R}, \mathcal{S})$ denotes the class of cones of inflations from objects in $\mathcal{R}$ to objects in $\mathcal{S}$.

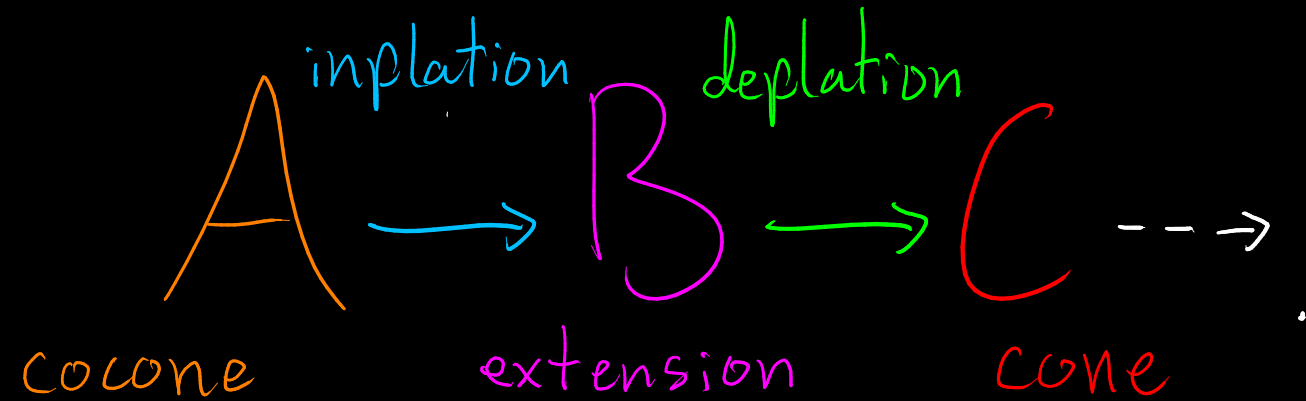
Rmk. In an extriangulated category $(\mathcal{C}, \mathbb{E}, s)$, the common generalization of kernel-cokernel pairs and distinguished triangles are called s -conflations, and they are denoted by

$$\begin{array}{c} \text{inflation} \quad \text{deflation} \\ A \xrightarrow{\quad} B \xrightarrow{\quad} C \dashrightarrow \\ \text{cocone} \qquad \qquad \text{cone} \end{array}$$

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- (a) $\text{Cone}(\mathcal{R}, \mathcal{S})$ denotes the class of **cones** of **inflations** from objects in $\mathcal{R}$ to objects in $\mathcal{S}$.
- (b) $\text{Cocone}(\mathcal{R}, \mathcal{S})$ denotes the class of **cocones** of **deflations** from objects in $\mathcal{R}$ to objects in $\mathcal{S}$.

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- (c) $\mathcal{R} * \mathcal{S}$ denotes the class of **extensions** of objects in $\mathcal{S}$ by objects in $\mathcal{R}$.

Step. Let $(\mathcal{C}, E, s)$ be extriangulated and assume its subcategories are full, strict and additive.

Defn. For $S \subseteq \mathcal{C}$, let $\beta(S) := \{S' \in S \mid \exists S \rightarrow S' \rightarrow S'' \dashrightarrow, S' \in S \Rightarrow S'' \in S\}$ and $i(S) := \{C \in \mathcal{C} \mid \exists C \rightarrow S \rightarrow C' \dashrightarrow \text{ with } S \in S\}$.

Rmk. Let $S \subseteq \mathcal{C}$.

- (1) $\beta(S)$ is the biggest subcategory contained in S and closed under cones.
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Defn. We say $S \in \mathcal{C}$ is

- (a) *torsionfree* if $S * S \in S$ and $i(S) \subseteq S$;

Stp. Let $(\mathcal{C}, E, s)$ be extriangulated and assume its subcategories are full, strict and additive.

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Dfn. We say $\mathcal{S} \subseteq \mathcal{C}$ is

- (a) *torsionfree* if $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$ and $i(\mathcal{S}) \subseteq \mathcal{S}$;
- (b) *thick* if $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$, $\text{smd}(\mathcal{S}) \subseteq \mathcal{S}$, $\text{Cone}(\mathcal{S}, \mathcal{S}) \subseteq \mathcal{S}$ and $\mathcal{S} = \text{Cocone}(\mathcal{S}, \mathcal{S})$;

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- (c) *resolving* if $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$, $\text{smd}(\mathcal{S}) \subseteq \mathcal{S}$, $\mathcal{S} \cong \text{Cocone}(\mathcal{S}, \mathcal{S})$ and $\text{Cone}(\mathcal{C}, \mathcal{S}) \cong \mathcal{C}$.

Stp. Suppose $(\mathcal{C}, \mathbb{E}, s)$ is hereditary.

Prp. Let $S \in \mathcal{C}$.

(a) If $S * S \subseteq S$, then $i(S)$ is the smallest torsionfree class containing S .

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Rmk. If $(X, Y) \in \text{cot}(\mathcal{C})$, then X is torsionfree.

Stp. Suppose $(\mathcal{C}, \mathbb{E}, s)$ is reduced O-Auslander.

Dfn. Let $S \in \mathcal{C}$ be such that $S^*S \in \mathbb{E}$. Then S has enough injectives if the induced extriangulated category $(S, \mathbb{E}|_S, s|_S)$ has enough $(\mathbb{E}|_S-)$ injectives (see [NP19]).

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Prp. [Sak] The maps $\{S \in \text{torp}(\mathcal{C}) \mid S \text{ has enough injectives}\} \xrightleftharpoons[G_1]{F_1} \text{c-cot}(\mathcal{C})$ given by $F_1(S) := (S, S^{\perp 1})$ and $G_1(X, Y) := X$ are mutually inverse.

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Qst. Does the previous bijection restrict to subcategories with enough injectives?

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Rmk. [Gar23, 3.10] A thick subcategory $\mathcal{R}$ of a hereditary extriangulated category $\mathcal{C}$ has enough injectives if, and only if, $\mathcal{R} = \text{thick}(\mathcal{U})$ for some presilting subcategory $\mathcal{U} \subseteq \mathcal{C}$ such that $\text{Cone}(\mathcal{U}, \mathcal{U}) \subseteq \mathcal{U}$.

Prop. Let $(\chi, \gamma) \in \text{Cosilt-cot}(\mathcal{K}^{[0,1]}(\text{Inj } \Lambda))$ for Λ a f.d. $\bar{k}$ -alg. Then there exist $\S$ -conflations

$$\omega_1 \rightarrow \omega_0 \xrightarrow{\psi} D(\Lambda) \dashrightarrow \quad \text{and} \quad D(\Lambda)[-1] \xrightarrow{\phi} \omega_1 \rightarrow \omega_0 \dashrightarrow$$

such that

(a) $\text{Prod}(\omega_0 \oplus \omega_1) = \chi \cap \gamma$;

(b) ψ is a minimal right $\chi \cap \gamma$ -approximation of $D(\Lambda)$ (+ dual statement).

Prop. Let $(\mathcal{X}, \mathcal{Y}) \in \text{Cosilt-cot}(\mathcal{K}^{[0,1]}(\text{Inj } \Lambda))$ for Λ a f.d. $\bar{k}$ -alg. Then there exist $\S$ -complations

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Moreover, $i(\beta(\mathcal{X})) = \mathcal{X}$. In particular, $\mathcal{X}$ is thickly generated.

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Rmk. In the silting case, the minimality of the approximations – and, thus, the equality – do not hold in general.

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Moreover, $i(\beta(X)) = X$. In particular, X is thickly generated.

Rmk. In the silting case, the minimality of the approximations – and, thus, the equality – do not hold in general.

Qst. Is there an “intrinsic” characterization for the subcategories $S \subseteq \mathcal{K}_{\mathbb{I}}^2$ such that $S = \beta(X)$ for $(X, Y) \in \text{Cosilt}(\mathcal{K}_{\mathbb{I}}^2)$?

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