Silting-discreteness of group algebras

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Perspectives on Tilting Theory and Related Topics

Notation

- k : algebraically closed field
- Λ : finite dimensional k-algebra
- $\operatorname{mod}\Lambda$: the category of right Λ-modules of finite dimension

§. τ-Tilting finiteness and silting-discreteness

Def.-Prop. [Demonet-Iyama-Jasso '19]

Λ is τ-tilting finite

$:\Leftrightarrow \# \, s\tau\text{-tilt}\,\Lambda < \infty$.

$\Leftrightarrow \# \, \text{brick}\,\Lambda < \infty$.

$\Leftrightarrow$ Every torsion class in $\operatorname{mod}\Lambda$ is functorially finite.

Rmk.

- $\begin{cases} \Lambda \xrightarrow{\exists} \!\!\!\rightarrow \Gamma : \text{surj. alg. hom.} \\ \Gamma : \tau\text{-tilting infinite} \end{cases} \Rightarrow \Lambda : \tau\text{-tilting infinite}.$

- $\Lambda := kQ$ (Q : acyclic quiver) : τ-tilting finite $\overset{\text{iff}}{\Longleftrightarrow}$ Q : Dynkin.

- We can show τ-tilt. inf. by the shape of quivers in some cases.

e.g.) $\Lambda := k\left[\begin{tikzcd} \bullet \arrow[r, shift left] \arrow[d, shift left] & \bullet \arrow[l, shift left] \arrow[d, shift left] \\ \bullet \arrow[u, shift left] \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[u, shift left] \end{tikzcd}\right] \Big/ \sim \twoheadrightarrow k\left[\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \bullet \arrow[l] \arrow[u] \end{tikzcd}\right]$: τ-tilting infinite

arbitrary (admissible) relation

$\therefore\ \Lambda : \tau\text{-tilting infinite}$

Def. Λ is *silting-discrete*

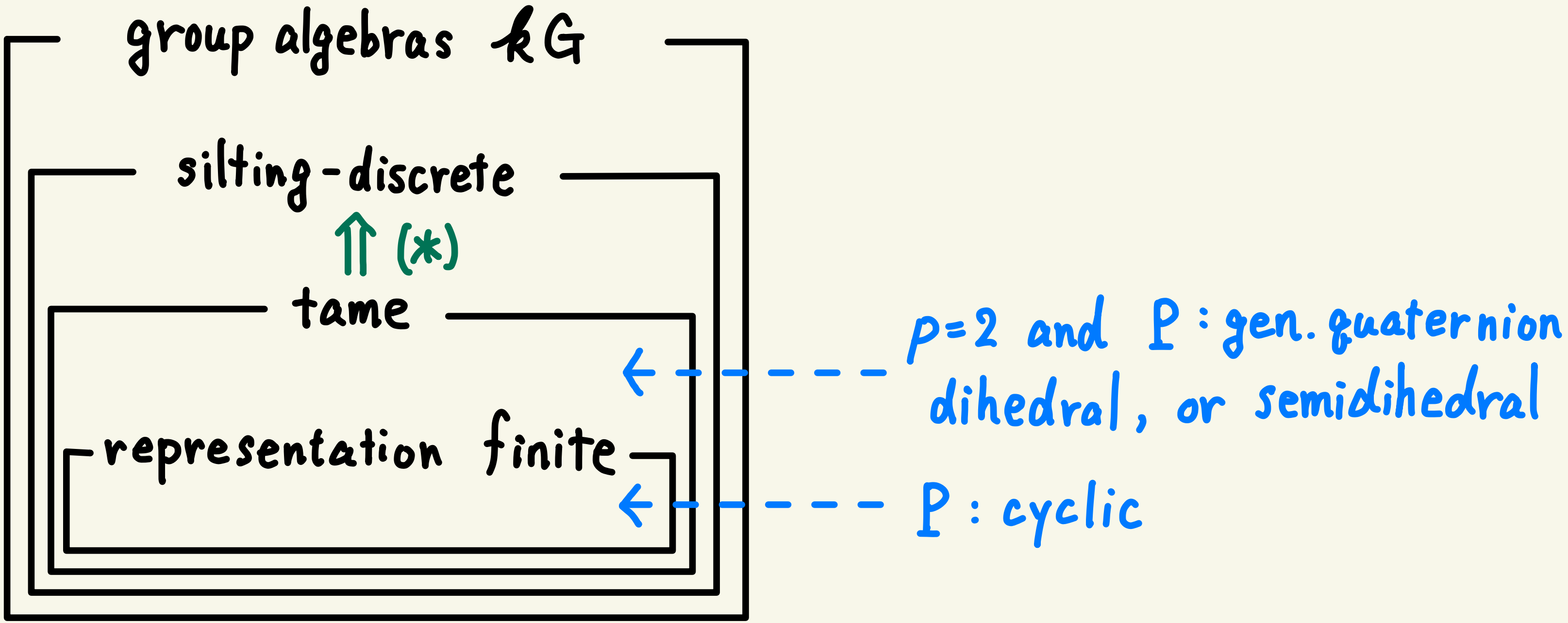
$:\Leftrightarrow {}^{\forall}S, T \in \operatorname{silt}\Lambda$ with $S \geq T$, $\#\{U \in \operatorname{silt}\Lambda \mid S \geq U \geq T\} < \infty$.

Rmk.
- Λ: silting-discrete $\Rightarrow$ Λ: τ-tilting finite.
- If Λ is silting-discrete, then the silting quiver of Λ is (weakly) connected.

Prop. [Aihara-Mizuno '17] Assume Λ is symmetric. Then Λ is silting-discrete iff every algebra derived equivalent to Λ is τ-tilting finite.

§. Silting discreteness of group algebras

$p = \operatorname{char} k > 0$, G : finite group, P : Sylow p-subgrp. of G.



Reasons for (*)

1. Representation finite symmetric algebras are silting-discrete by [Aihara '13].

2. Algebras of dihedral, semidihedral, or quaternion type
   the class invariant under derived equivalences containing (rep. inf.) tame blocks of group algebras
   are τ-tilting finite by [Eisele-Janssens-Raedschelders '18], and hence silting-discrete.

Question What structure controls silting-discreteness of kG?

Rmk. Silting-discreteness of a group algebra kG is NOT determined by its Sylow p-subgroup P.

e.g.) $k[C_p \times C_p]$: silting-discrete.

$k[(C_p \times C_p) \rtimes C_2]$: not silting-discrete for $^\forall p \neq 2$.

sending to the inverse

Def. We call $P \cap O^p(G)$ a p-hyperfocal subgroup of G.

the smallest normal subgrp. of G s.t. its quotient is a p-group

	G	P	$O^p(G)$	$P \cap O^p(G)$
e.g.)	$C_p \times C_p$	$C_p \times C_p$	1	1
		$\parallel$		$\neq$
	$(C_p \times C_p) \rtimes C_2$	$C_p \times C_p$	$(C_p \times C_p) \rtimes C_2$	$C_p \times C_p$

$R := P \cap O^p(G)$: a p-hyperfocal subgroup of G

Prop. [Kimura-Koshio-Kozakai-Minamoto-Mizuno '25]
Assume $N \trianglelefteq G$ and G/N is a p-group.
Then kN : silting-discrete $\Rightarrow$ kG : silting-discrete.

Cor. kG is silting-discrete if one of the following holds:
(a) R is cyclic.
(b) $p=2$ and R is dih., semidih., or gen. quat.

$\because$ Since R is a Sylow p-subgrp. of $O^p(G)$,
(a) or (b) $\Rightarrow$ $kO^p(G)$: tame $\Rightarrow$ $kO^p(G)$: silt.-discr. $\Rightarrow$ kG : silt.-discr.

Our conjecture The converse of Cor. holds.

If P is abelian and we assume that Broué's abelian defect conjecture is true, then our conjecture can be reduced to the case $G = P \rtimes H$ (P : abelian p-group, $\underbrace{H : p'\text{-group}}_{p \nmid \# H}$).

Broué's abelian defect conjecture If P is abelian, then the principal blocks $B_0(kG)$ and $B_0(kN_G(P))$ are derived equivalent.

Rmk. By the Schur-Zassenhaus theorem,

$$\exists H : p'\text{-group} \quad \text{s.t.} \quad N_G(P) = P \rtimes H.$$

Thm.1 [H-Kozakai] P : abelian p-group,

H: abelian p'-group acting on P, $G := P \rtimes H$.

Then kG is ~~τ-tilt. fin.~~ iff one of the following holds:

silt.-discr.

(a) $p = 2$ and R is trivial or $C_2 \times C_2$.

(b) $p \geq 3$ and R is cyclic.

Thm.2 [H] Let H be a p'-subgrp. of $\mathfrak{S}_n$ and $G := (C_{p^\ell})^n \rtimes H$.

Assume $p^\ell \geq n$. Then kG is ~~τ-tilt. fin.~~ iff R is cyclic.

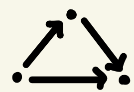
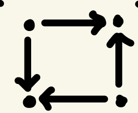
silt.-discr.

§. Sketch of proof of Thm. 1

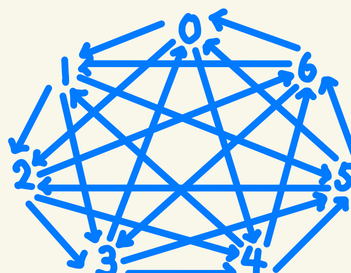
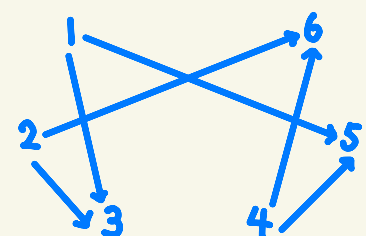
$G := P \rtimes H \quad \left(\begin{array}{l} P : \text{abelian } p\text{-grp.} \\ H : \text{abelian } p'\text{-grp.} \end{array} \right)$

We know the quiver and relations for kG.

Then we can take τ-tilt. inf. quotient algebras of kG

such as $k[\cdot \rightrightarrows \cdot]$, k[], k[], $\cdots$.

e.g.) • $k[(C_p \times C_p) \rtimes C_2] \cong k[\cdot \overset{\rightrightarrows}{\leftleftarrows} \cdot]/\sim \;\twoheadrightarrow\; k[\cdot \rightrightarrows \cdot]$ $(p \geq 3)$

• $k[(C_2)^3 \rtimes C_7] \cong k$[]$/\sim \;\twoheadrightarrow\; k$[].

$(p = 2)$

- $p=2$, $G := \underbrace{(C_{2^\ell})^2}_{\langle a\rangle \times \langle b\rangle} \rtimes \underbrace{C_3}_{\langle c\rangle}$

$$c : a \mapsto b \mapsto a^{-1}b^{-1}$$

$$kG \cong \frac{k\left[\begin{tikzcd} & 2 \arrow[dr, bend left, "\alpha"] \arrow[dl, "\beta"'] & \\ 1 \arrow[ur, bend left, "\alpha"] \arrow[rr, "\beta"] & & 3 \arrow[ul, "\beta"'] \arrow[ll, bend left, "\alpha"] \end{tikzcd}\right]}{\left(\begin{matrix} \alpha\beta-\beta\alpha \\ \alpha^{2^\ell}, \beta^{2^\ell} \end{matrix}\right)} \twoheadrightarrow \begin{cases} k\left[\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[rr] & & 3 \end{tikzcd}\right] & (\ell \geq 2) : \tau\text{-tilt. inf.} \\ \dfrac{k\left[\begin{tikzcd} & 2 \arrow[dr, "\alpha_2"] & \\ 1 \arrow[ur, "\alpha_1"] \arrow[rr] & & 3 \end{tikzcd}\right]}{(\alpha_2\alpha_1)} & (\ell = 1) : \tau\text{-tilt. fin.} \end{cases}$$

§. Sketch of the proof of Thm. 2

$G := (C_{p^\ell})^n \rtimes H \quad \left(\begin{array}{l} H \leq \mathfrak{S}_n \\ p^\ell \geq n \end{array} \right)$

$$kG \overset{\exists}{\twoheadrightarrow} \underbrace{k[x_1, \cdots, x_n]/(\text{Sym. poly. of deg.} > 0) \rtimes H}_{\parallel}$$

Γ : selfinjective

We can compute the Cartan matrix of Γ.

By appying Prop. in the next slide to Γ,

we can show that kG is τ-tilting infinite.

Λ : selfinjective algebra

$P_1, \cdots, P_t$: all indec. proj. Λ-modules

$\nu \in \mathfrak{S}_t$: Nakayama permutation (i.e. $P_i \cong P_{\nu(i)} \underset{\Lambda}{\otimes} D\Lambda$)

C_Λ : Cartan matrix of Λ (i.e. $(C_\Lambda)_{ij} = \dim \mathrm{Hom}_\Lambda(P_i, P_j)$)

$\boxed{\begin{array}{l}\underline{\text{Prop.}}\ [\mathrm{H}]\ \text{If } {}^\exists v \in \mathbb{Z}^t \setminus \{0\} \text{ s.t. } v^T C_\Lambda v \leq 0 \text{ and } \underbrace{\nu \cdot v}_{\nu \text{ permutes entries of } v} = v, \\ \text{then } \Lambda : \tau\text{-tilting infinite.}\end{array}}$

e.g.) $k[(C_p)^3 \rtimes \mathfrak{S}_3] \twoheadrightarrow \Gamma \underset{\text{Morita}}{\sim} k[\,1 \rightleftarrows 2 \rightleftarrows 3\,]/\sim \ \cong \begin{smallmatrix}1\\2\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & 2 & 3\\ 1 & 2 & 3\\ & 2 & \end{smallmatrix} \oplus \begin{smallmatrix}3\\2\\2\\1\end{smallmatrix}.$

$(p > 3)$

$\nu = (1\,3)$, $C_\Gamma = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix} \rightsquigarrow v = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ satisfies the assumption.

$\therefore\ \Gamma : \tau$-tilting infinite.