§1 Invariant Theory

K algebraically closed field

G reductive linear algebraic group $/K$

e.g., $G = GL_n, O_n, SL_n, Sp_n$, finite, $\mathbb{G}_m = GL_1 = (K^\times, \cdot)$ mult. group

and products (e.g. r-dim torus $\mathbb{G}_m^r$)

not: $\mathbb{G}_a = (K,+) = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \middle| t \in K \right\}$

X affine G-variety, i.e., $G \times X \to X$ regular action

$K[X]$ affine coordinate ring, $X = \operatorname{Spec} K[X]$

G acts on $K[X]$ by automorphisms:

if $f \in K[X]$, $g \in G$ then $g \cdot f$ defined by

$(g \cdot f)(x) = f(g^{-1} \cdot x)$, $x \in X$.

def. $K[X]^G = \{ f \in K[X] \mid \forall g \in G \ g \cdot f = f \}$ invariant ring

Theorem (Hilbert, Nagata, Haboush)
$K[X]^G$ is finitely generated over K.

$X/\!/G = \operatorname{Spec} K[X]^G$ is affine variety.
inclusion $K[X]^G \hookrightarrow K[X]$ corresponds to morphism $\pi : X \to X/\!/G$.

Properties:
1. π is surjective
2. $\pi(g\cdot x) = \pi(x)$ for $x \in X$, $g \in G$.
3. $\pi(x) = \pi(y) \iff \overline{G\cdot x} \cap \overline{G\cdot y} \neq \emptyset$

if all G orbits are closed in X then
$\pi(x) = \pi(y) \iff G\cdot x = G\cdot y$.
"geometric quotient"

Example:

$V = K^n$, $G = GL(V) = GL_n(K)$

$X = (V^*)^p \oplus V^q$

Define $\pi_{ij} : X \to K$ by $\pi_{ij}(f_1, f_2, \ldots, f_p, v_1, v_2, \ldots, v_q) = f_i(v_j)$

FFT of Inv. Theory: $K[X]^G = K[\pi_{ij},\ 1 \leq i \leq p,\ 1 \leq j \leq q]$

Let $U = K^q$, $W = K^p$, $X = \mathrm{Hom}(V, W) \oplus \mathrm{Hom}(U, V)$

$$\pi = (\pi_{ij}) : X \longrightarrow X/\!/G = \mathrm{Hom}^{(n)}(U, W)$$

$$\pi(A, B) = AB \qquad\qquad \mathrm{Hom}^{(n)}(U,W) = \{A \in \mathrm{Hom}(U, W) \mid \mathrm{rk}\, A \leq n\}$$

§2 Geometric Invariant Theory

V representation of G

$X \subseteq V$ G-invariant cone, $\pi : X \to X/\!/G$ quot.

$\mathbb{P}(X) = \operatorname{Proj} K[X] \subseteq \mathbb{P}(V)$ $\qquad \pi(0) = 0$

$\mathbb{P}(X/\!/G) = \operatorname{Proj} K[X]^G$

$$
\begin{tikzcd}
\mathbb{P}(X) \arrow[r, dashed, "\text{rational}"] & \mathbb{P}(X/\!/G) \\
\mathbb{P}(X^{ss}) \arrow[u, phantom, "\text{open}\ \cup"] \arrow[ur, "\tilde{\pi}", "\text{quot.}"'] & \\
\mathbb{P}(X^{s}) \arrow[u, phantom, "\text{open}\ \cup"] \arrow[uur, "\text{geom. quot.}"'] &
\end{tikzcd}
$$

$\mathcal{N} = \pi^{-1}(0)$ null cone, $X^{ss} = X \setminus \mathcal{N}$ semi-stable points

$X^{s} = \{ x \in X^{ss} \mid G\cdot x \text{ closed, and } \dim G = \dim G\cdot x \}$ stable points

(if G acts faithfully)

Example:

$V = K^n$, $X = \mathrm{End}(V)$, $G = GL(V)$

$A \in X$, char. polyn.:

$$\chi_A(t) = \det(tI - A) = t^n - e_1(A)t^{n-1} + e_2(A)t^{n-2} - \cdots + (-1)^n e_n(A)$$

$$K[X]^G = K[e_1, e_2, \ldots, e_n]$$

$$\mathcal{N} = \{A \in X \mid e_1(A) = \cdots = e_n(A) = 0\}$$

$A \in \mathcal{N} \Leftrightarrow A$ nilpotent

$$X^{ss} = X \setminus \mathcal{N}$$

orbit of $A \in X$ closed $\Leftrightarrow$ A is diagonalizable (semi-simple)

§3. Representation Spaces

$Q = (\underset{\text{vertices}}{Q_0}, \underset{\text{arrows}}{Q_1}, h, t)$

$h, t : Q_1 \to Q_0$ head, tail

A reps. V of Q is:

$V(x)$, $x \in Q_0$ fin. dim. K-vector spaces together with linear maps $V(a) : V(ta) \to V(ha)$, $a \in Q_1$.

$\underline{\dim} V \in \mathbb{N}^{Q_0}$, $(\underline{\dim} V)(x) = \dim V(x)$ dim vector

if $\alpha \in \mathbb{N}^{Q_0}$ and we chose bases in $V(x)$, $x \in Q_0$ then $V \in \operatorname{Rep}_\alpha(Q) = \bigoplus_{a \in Q_1} \operatorname{Hom}(K^{\alpha(ta)}, K^{\alpha(ha)})$

Representation Space

$GL_\alpha = \prod_{x \in Q_0} GL_{\alpha(x)}(K)$ acts on $Rep_\alpha(Q)$ by

base change:

if $g = (g(x),\ x \in Q_0) \in GL_\alpha$
$V = (V(a),\ a \in Q_1) \in Rep_\alpha(Q)$
then $g \cdot V = (g(ha)V(a)g(ta)^{-1},\ a \in Q_1)$

Bijection:

GL_α-orbits in $Rep_\alpha(Q)$ $\longleftrightarrow$ isomorphism classes of α-dimensional representations

S_x simple representation at x
$\delta_x = \underline{\dim}\, S_x$ $\quad \bigoplus_{x \in Q_0} S_x^{\alpha(x)}$ is $0 \in Rep_\alpha(Q)$

$Rep(Q)$ = category of fin. dim. representations of Q.

§4. Invariants for quivers (LeBruyn-Procesi, Donkin)

$$I(Q,\alpha) = K[\mathrm{Rep}_\alpha(Q)]^{GL_\alpha}$$

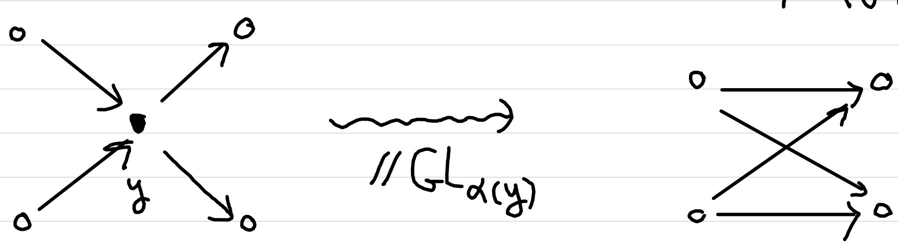
Theorem:

char $K=0$ (LeBruyn-Procesi): $I(Q,\alpha)$ generated by $\mathrm{Tr}(V(p))$, p cyclic path

char K arbitrary (Donkin): $I(Q,\alpha)$ generated by coeffs of $\chi_{V(p)}(t)$, p cyclic path.

idea:

One can reduce to case $|Q_0|=1$ by FFT of IT



If $V \in \mathrm{Rep}_\alpha(Q)$ then:

$V \in \mathcal{N} \iff$ there exists a filtration
$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_s = V$ such that
$\forall i \ \exists x \ \ V_i / V_{i-1} \cong S_x$

$\iff$ V is nilpotent representation.

V is stable $\iff$ V is simple.

$GL_\alpha \cdot V$ is closed $\iff$ V is semi-simple.

Q acyclic $\Rightarrow$ $I(Q, \alpha) = K$

§5 GIT for Quivers (after A. King)

Q acyclic quiver

$\sigma \in \mathbb{Z}^{Q_0}$ weight

if $\alpha \in \mathbb{N}^{Q_0}$ then $\sigma(\alpha) := \sum_{x \in Q_0} \sigma(x)\alpha(x)$

$\chi_\sigma : GL_\alpha \to \mathbb{G}_m = (K^\times, \cdot)$

$(g(x), x \in Q_0) \mapsto \prod_{x \in Q_0} (\det g(x))^{\sigma(x)}$

$SI(Q,\alpha)_\sigma = \{ f \in K[\mathrm{Rep}_\alpha(Q)] \mid \forall g \in GL_\alpha \; g \cdot f = \chi_\sigma(g) f \}$

space of semiinvariants of weight σ

$\lambda \in K^\times$, $g_\lambda = (\lambda I_{\alpha(x)}, x \in Q_0)$ acts trivially on $\mathrm{Rep}_\alpha(Q)$

if $0 \neq f \in SI(Q,\alpha)_\sigma$ then $f = g_\lambda \cdot f = \chi_\sigma(g_\lambda) f$

so $1 = \chi_\sigma(g_\lambda) = \lambda^{\sigma(\alpha)}$ so $\sigma(\alpha) = 0$

view χ_σ as 1-dim. representation

$$\pi: \operatorname{Rep}_\alpha(Q) \oplus \chi_\sigma \longrightarrow (\operatorname{Rep}_\alpha(Q) \oplus \chi_\sigma) /\!/ GL_\alpha \simeq \operatorname{Spec} SI(Q,\alpha,\sigma)$$

where $SI(Q,\alpha,\sigma) = \bigoplus_{n \geq 0} SI(Q,\alpha)_{n\sigma}$

$$\begin{array}{ccl} \operatorname{Rep}_\alpha(Q) & \subseteq & \mathbb{P}(\operatorname{Rep}_\alpha(Q) \oplus \chi_\sigma) \\ \text{\rotatebox{90}{$\subseteq$}} & & \\ \operatorname{Rep}_\alpha(Q)^{ss}_\sigma & = & \mathbb{P}((\operatorname{Rep}_\alpha(Q) \oplus \chi_\sigma)^{ss}) \end{array}$$

σ-semistable points

$$\begin{tikzcd} \operatorname{Rep}_\alpha(Q)^{ss}_\sigma \arrow[r, "\pi_\sigma", "\text{good quot.}"'] & \operatorname{Rep}_\alpha(Q) /\!/_\sigma GL_\alpha = \operatorname{Proj} SI(Q,\alpha,\sigma) \\ \operatorname{Rep}_\alpha(Q)^{s}_\sigma \arrow[u, phantom, "\text{\rotatebox{90}{$\subseteq$}}"] \arrow[ur, "\text{geom. quot.}"'] & \end{tikzcd}$$

King's Criterion:

if $\sigma(\alpha)=0$, $V \in \mathrm{Rep}_\alpha(Q)$ then

V is σ-semistable $\Leftrightarrow$ $\sigma(\underline{\dim} W) \leq 0$ for every subrepr. $W \subseteq V$

V is σ-stable $\Leftrightarrow$ $\sigma(\underline{\dim} W) < 0$ for every subrep. $0 \neq W \subsetneq V$

$$SI(Q,\alpha) = \bigoplus_{\sigma \in \mathbb{Z}^{Q_0}} SI(Q,\alpha)_\sigma = K[\mathrm{Rep}_\alpha(Q)]^{SL_\alpha}$$

where $SL_\alpha = \prod_{x \in Q_0} SL_{\alpha(x)}(K) \subset GL_\alpha$

§6 Semi-Invariants

$\alpha, \beta \in \mathbb{N}^{Q_0}$, $\quad \langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha)$

Euler form

P_x projective at $x \in Q_0$.

can. resolution:

$$0 \longrightarrow \bigoplus_{a \in Q_1} V(a) \otimes P_{ha} \longrightarrow \bigoplus_{x \in Q_0} V(x) \otimes P_x \longrightarrow V \longrightarrow 0$$

Apply $\mathrm{Hom}_Q(-, W)$: $\quad V \in \mathrm{Rep}_\alpha(Q), W \in \mathrm{Rep}_\beta(Q)$

$$0 \to \mathrm{Hom}_Q(V, W) \to \bigoplus_{x \in Q_0} \mathrm{Hom}(V(x), W(x)) \xrightarrow{d_W^V} \bigoplus_{a \in Q_1} \mathrm{Hom}(V(ta), W(ha)) \to \mathrm{Ext}_Q^1(V, W) \to 0$$

$\langle \alpha, \beta \rangle = \dim \mathrm{Hom}_Q(V, W) - \dim \mathrm{Ext}(V, W)$

$\langle \alpha, \beta \rangle = 0 \iff d_W^V$ is square matrix define $c(V,W) = c^V(W) = c_W(V) = \det d^V_W$

Schofield: $c^V \in SI(Q,\beta)_{\langle\alpha,\cdot\rangle}$

$c_W \in SI(Q,\alpha)_{-\langle\cdot,\beta\rangle}$

($\langle\alpha,\cdot\rangle$ functional on dim vecs, weight)

Now $c(V,W) \neq 0 \iff \mathrm{Hom}_Q(V,W) = 0 \iff \mathrm{Ext}_Q(V,W) = 0$

Theorem (D-Weyman):

$SI(Q,\beta)_{\langle\alpha,\cdot\rangle}$ spanned by c^V, $V \in \mathrm{Rep}_\alpha(Q)$

$SI(Q,\alpha)_{-\langle\cdot,\beta\rangle}$ ——— c_W, $W \in \mathrm{Rep}_\beta(Q)$

$SI(Q,\beta)_{\langle\alpha,\cdot\rangle} \cong SI(Q,\alpha)^*_{-\langle\cdot,\beta\rangle}$

if $0 \to V_1 \to V \to V_2 \to 0$ exact, $\langle \underline{\dim} V_1, \beta\rangle = 0$ then $c^V = c^{V_1} c^{V_2}$.

def: $\alpha \circ \beta = \dim SI(Q,\beta)_{\langle \alpha, \cdot \rangle}$

Example: $Q =$
$$\begin{tikzcd} \circ \arrow[r, "a"] & \circ & \circ \arrow[l, "b"'] \\ & \circ \arrow[u, "c"] & \end{tikzcd}$$

$$\beta = \begin{array}{ccc} n & 2n & n \\ & n & \end{array}$$

V_1, V_2, V_3 undecomposables of dim

$\alpha_1 = \begin{array}{ccc} 1 & 1 & 1 \\ & 0 & \end{array}$, $\alpha_2 = \begin{array}{ccc} 1 & 1 & 0 \\ & 1 & \end{array}$, $\alpha_3 = \begin{array}{ccc} 0 & 1 & 1 \\ & 1 & \end{array}$, $\langle \alpha_1, \cdot \rangle = \begin{array}{ccc} 1 & -1 & 1 \\ & 0 & \end{array}$

$$c^{V_1}(W) = \det \begin{bmatrix} W(a) & W(b) \end{bmatrix}$$

$$SI(Q,\beta) = K[c^{V_1}, c^{V_2}, c^{V_3}]$$

Domokos - Zubkov:

$\sigma = \sigma_+ - \sigma_-$ $\quad\quad\quad$ $\sigma \in \mathbb{Z}^{Q_0}$, $\quad \sigma_+, \sigma_- \in \mathbb{N}^{Q_0}$, $\quad \sigma(\alpha) = 0$

$x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q \in Q_0$

$z \in Q_0$ appears $\sigma_+(z)$ times among x's

$\sigma_-(z)$ ———————— y's

P_{ij} linear combination of paths from x_j to y_i.

then we have semi-invariant

$$V \in \mathrm{Rep}_\alpha(Q) \longmapsto \det \left(V(P_{ij}) \right)_{i,j}$$

such semi-invariants span $SI(Q, \alpha)$

§7 Root Systems

Q quiver

$\alpha \in \mathbb{N}^{Q_0}$ called (positive) root if there exists an undecomposable representation of dim α.

$\delta_x = \underline{\dim}\, S_x$, $x \in Q_0$ simple roots

$\Phi^+ =$ set of positive roots

$\Phi^- = -\Phi^+$, $\Phi = \Phi^+ \cup \Phi^-$

$Q^0 =$ Q without orientation of arrows

Gabriel's Theorem:

Q finite type $\Longleftrightarrow$ Q^0 union of Dynkin graphs of type A, D, E.

Φ is root system of semi-simple Lie algebra of type Q^0.

Kac generalized to arbitrary Q.

$(\alpha,\beta) := \langle\alpha,\beta\rangle + \langle\beta,\alpha\rangle$

$\sigma_x : \mathbb{Z}^{Q_0} \longrightarrow \mathbb{Z}^{Q_0}$

$\sigma_x(\alpha) = \alpha - (\alpha,\delta_x)\,\delta_x$

$W = \langle \sigma_x \mid x \in Q_0 \rangle$ Weyl group

$\Phi_{re} = \bigcup_{x\in Q_0} W\delta_x$ real roots

$\Phi_{re} = \Phi_{re}^+ \cup \Phi_{re}^-$ $\quad \Phi_{re}^+ = \Phi_{re} \cap \mathbb{N}^{Q_0}$, $\Phi_{re}^- = -\Phi_{re}^+$

let $K = \{\alpha \in \mathbb{N}^{Q_0} \mid (\alpha,\delta_x) \leq 0$ for all $x \in Q_0$ and support of α has 1 connected component$\}$

$\Phi_{im}^+ = WK$, $\Phi_{im}^- = -\Phi_{im}^+$ $\quad \Phi_{im} = \Phi_{im}^+ \cup \Phi_{im}^-$

Kac: $\Phi^+ = \Phi_{re}^+ \cup \Phi_{im}^+$, Φ Root system of Kac moody Lie algebra of type Q^0.

note $(w(\alpha), w(\beta)) = (\alpha, \beta)$ for all $w \in W$

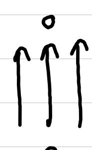
if $\alpha \in \Phi^+_{re}$, then $\langle \alpha, \alpha \rangle = 1$

if $\alpha \in \Phi^+_{im}$ then $\langle \alpha, \alpha \rangle \leq 0$.

α called isotropic if $\langle \alpha, \alpha \rangle = 0$.

If $\alpha \in \Phi^+_{re}$ then there is unique undecomposable representation of dim α.

if $\alpha \in \Phi^+_{im}$ then there is d-dimensional "family" of undecomposable α-dimensional repr. where $d = 1 - \langle \alpha, \alpha \rangle$.

Example: $Q =$ 

$$\Phi^+_{re} = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}8\\3\end{pmatrix}, \begin{pmatrix}21\\8\end{pmatrix}, \ldots, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}3\\8\end{pmatrix}, \begin{pmatrix}8\\21\end{pmatrix}, \ldots \right\}$$

$$\Phi^+_{im} = \left\{ \begin{pmatrix}a\\b\end{pmatrix} \,\middle|\, \begin{matrix} a^2 - 3ab + b^2 \leq 0 \\ a, b > 0 \end{matrix} \right\} = \left\{ \begin{pmatrix}a\\b\end{pmatrix} \,\middle|\, a, b > 0 \ \frac{3-\sqrt{5}}{2} < \frac{a}{b} < \frac{3+\sqrt{5}}{2} \right\}$$

§8 Canonical decomposition

$V \in \mathrm{Rep}_\alpha(Q)$

$(GL_\alpha)_V \cong \mathrm{Hom}_Q(V,V)^\times$ stabilizer

$GL_\alpha \cdot V \subseteq \mathrm{Rep}_\alpha(Q)$ orbit, open in its closure

$N_V(GL_\alpha \cdot V) \cong \mathrm{Ext}_Q(V,V)$ normal space to orbit

$GL_\alpha \cdot V \subseteq \mathrm{Rep}_\alpha(Q)$ dense $\Leftrightarrow$ $GL_\alpha \cdot V$ open

$\Updownarrow$

V is partial tilting $\Leftrightarrow$ $\mathrm{Ext}_Q(V,V)$

V is a <u>Schur representation</u> or <u>brick</u> if $\mathrm{Hom}_Q(V,V) = K$

if V is Schur then V is indecomposable

α is <u>Schur root</u> if there exists Schur reps. $V \in \mathrm{Rep}_\alpha(Q)$

α is Schur root $\Leftrightarrow$ general $V \in \mathrm{Rep}_\alpha(Q)$ is Schur $\Leftrightarrow$ general $V \in \mathrm{Rep}_\alpha(Q)$ is indecomp.

Example:

$$Q = \begin{tikzcd} & \overset{y}{\circ} \arrow[dr, "b"] & \\ x\,\circ \arrow[ur, "a"] \arrow[rr, "c"'] & & \circ\, z \end{tikzcd}$$

$\alpha = \begin{smallmatrix} & 2 & \\ 1 & & 1 \end{smallmatrix}$

$\langle \alpha, \alpha \rangle = 1$

$$\begin{tikzcd} & K^2 \arrow[dr, "{[0\ 1]}"] & \\ K \arrow[ur, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] \arrow[rr, "{[1]}"'] & & K \end{tikzcd}$$

indecomp.

if $V \in \mathrm{Rep}_\alpha(Q)$ general, then $V(b)V(a) \neq 0$ and V has summand S_y

α is real root, but not Schur root.

$\beta = \begin{smallmatrix} & 2 & \\ 2 & & 2 \end{smallmatrix}$

$\langle \beta, \beta \rangle = 0$

$$\begin{tikzcd} & K^2 \arrow[dr, "{\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]}"] & \\ K^2 \arrow[ur, "{\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]}"] \arrow[rr, "{\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]}"'] & & K^2 \end{tikzcd}$$

indec,

β root, not Schur root.

$\hom_Q(\alpha,\beta) = \min\{\dim \mathrm{Hom}_Q(V,W) \mid V \in \mathrm{Rep}_\alpha(Q), W \in \mathrm{Rep}_\beta(Q)\}$
$\mathrm{ext}_Q(\alpha,\beta) =$ —— $\dim \mathrm{Ext}_Q(V,W)$ ——————

$\hom_Q(\alpha,\beta) = \dim \mathrm{Hom}_Q(V,W)$ for general
$\mathrm{ext}_Q(\alpha,\beta) = \dim \mathrm{Ext}_Q(V,W)$ $\quad (V,W) \in \mathrm{Rep}_\alpha(Q) \times \mathrm{Rep}_\beta(Q)$

$\langle \alpha,\beta \rangle = \hom_Q(\alpha,\beta) - \mathrm{ext}_Q(\alpha,\beta)$

Kac canonical decomposition:
if $\alpha \in \mathbb{N}^{Q_0}$ then there exists $s \geq 0, \alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{N}^{Q_0}$
such that a general $V \in \mathrm{Rep}_\alpha(Q)$ has a decomposition
$V \cong V_1 \oplus V_2 \oplus \cdots \oplus V_s$ where V_i indec. and $\underline{\dim} V_i = \alpha_i$.

we say: $\alpha = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_s$ is canonical decomp. of α

Theorem: $\alpha = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_s$ is can. dec.
$\iff$ $\alpha_1, \alpha_2, \ldots, \alpha_s$ are Schur roots, $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_s$, and
$\mathrm{ext}_Q(\alpha_i, \alpha_j) = 0$ for $i \neq j$.

Example: $Q = $

$$\begin{tikzcd} & \circ \arrow[dr] & \\ \circ \arrow[ur] \arrow[rr] & & \circ \end{tikzcd}$$

$$\boxed{\begin{smallmatrix} & 2 & \\ 1 & & 1 \end{smallmatrix}} = \boxed{\begin{smallmatrix} & 1 & \\ 1 & & 1 \end{smallmatrix}} \oplus \boxed{\begin{smallmatrix} & 1 & \\ 0 & & 0 \end{smallmatrix}}$$

$$\boxed{\begin{smallmatrix} & 2 & \\ 2 & & 2 \end{smallmatrix}} = \boxed{\begin{smallmatrix} & 1 & \\ 1 & & 1 \end{smallmatrix}} \oplus \boxed{\begin{smallmatrix} & 1 & \\ 1 & & 1 \end{smallmatrix}}$$

Example: $Q = $

$$\begin{tikzcd} \circ & \circ \arrow[l] & \circ \arrow[l, shift left] \arrow[l, shift right] \arrow[ll, bend left] \end{tikzcd}$$

$(6,33,17) = (5,27,14) \oplus (1,6,3)$, $\quad (12,66,34) = (10,54,28) \oplus (1,6,3)^{\oplus 2}$

D-Weyman: fast algorithm

Suppose α is Schur root, then:

if $\langle \alpha, \alpha \rangle \geq 0$ then $n\alpha = \alpha^{\oplus n} = \underbrace{\alpha \oplus \alpha \oplus \cdots \oplus \alpha}_{n}$ can. dec.

if $\langle \alpha, \alpha \rangle < 0$ then $n\alpha$ Schur root.

Suppose $\alpha = \alpha_1^{\oplus d_1} \oplus \alpha_2^{\oplus d_2} \oplus \cdots \oplus \alpha_s^{\oplus d_s}$ is can. decomp.
with $\alpha_1, \alpha_2, \ldots, \alpha_s$ distinct, $d_1, d_2, \ldots, d_s \geq 1$.

if $\langle \alpha_i, \alpha_i \rangle < 0$ then $d_i = 1$.

Schofield: after rearranging one may
assume $\hom_Q(\alpha_i, \alpha_j) = 0$ for $i<j$.

α is prehomogeneous
(Gl_α has dense orbit in $Rep_\alpha(Q)$) $\iff$ $\alpha_1, \alpha_2, \ldots, \alpha_s$ are <u>real</u> Schur roots.

$$\underset{\substack{\| \\ \dim \mathrm{Ext}_Q(V,V) \text{ for general } V.}}{\text{codim of general orbit}} = \sum_{i=1}^{s} d_i (1 - \langle \alpha_i, \alpha_i \rangle)$$

(may not be equal to $\mathrm{ext}_Q(\alpha, \alpha)$ because (V,V) not general)

$s \leq |Q_0|$

$s < |Q_0|$ if α not prehomogeneous

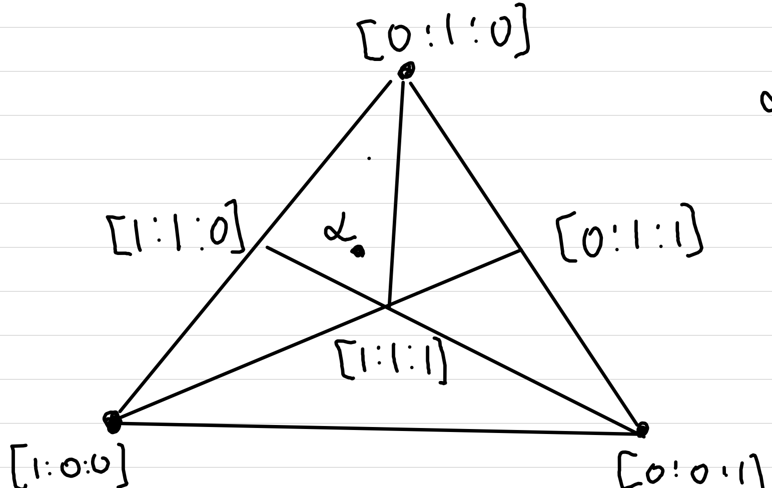
$\{ \alpha \in \mathbb{N}^{Q_0} \mid \alpha \text{ is prehomogeneous} \}$
form a simplicial cone

Example

$$\begin{tikzcd} & \circ \arrow[dr] & \\ \circ \arrow[ur] & & \circ \end{tikzcd}$$

finite type
every α is prehomogeneous

projective pic :



$\alpha = (3,4,2) =$
$(1,1,0) \oplus (0,1,0) \oplus (1,1,1)^{\oplus 2}$

§9 quiver Grassmannians

Suppose $\alpha = \beta + \gamma \qquad \alpha, \beta, \gamma \in \mathbb{N}^{Q_0}$

$$\mathrm{Grass}(\beta,\alpha) = \prod_{x \in Q_0} \mathrm{Grass}(\beta(x), \alpha(x))$$

$$\rotatebox{90}{$\in$}$$

$$R = (R(x) \subseteq K^{\alpha(x)}, \ x \in Q_0) \qquad \text{where } \dim R(x) = \beta(x)$$

$$Z(\beta,\alpha) = \{ (V,R) \in \mathrm{Rep}_\alpha(Q) \times \mathrm{Grass}(\beta,\alpha) \mid V(a)(R(ta)) \subseteq R(ha) \text{ for all } a \in Q_1 \}$$

$$\begin{tikzcd}
& \underset{\text{smooth}}{Z(\beta,\alpha)} \arrow[dl, "p"'] \arrow[dr, "q"] & \\
\mathrm{Rep}_\alpha(Q) & & \mathrm{Grass}(\beta,\alpha)
\end{tikzcd}$$

projective morphism (p) $\qquad$ vector bundle (q)

$p(Z(\beta,\alpha)) \subseteq \mathrm{Rep}_\alpha(Q)$ closed

$\beta \hookrightarrow \alpha$ means: a general α-dimensional representation has β-dim subrep.

$\alpha \twoheadrightarrow \gamma$ means: a general α-dim rep has γ-dim factor.

$\beta \hookrightarrow \alpha \iff \alpha \twoheadrightarrow \gamma \iff p$ dominant $\iff p$ surjective

$$\Updownarrow *$$

$\mathrm{ext}_Q(\beta,\gamma)=0$

* Theorem: Schofield, Crawley-Boevey (char $k>0$)

Suppose $\mathrm{ext}_Q(\beta,\gamma)=0$
and $V \in \mathrm{Rep}_\alpha(Q)$ general.

$Gr(\beta,V) := p^{-1}(V)$ smooth of dimension $\hom_Q(\beta,\gamma)$

We write $\beta \perp \gamma$ if $\hom_Q(\beta,\gamma) = \operatorname{ext}_Q(\beta,\gamma) = 0$.

$\beta \perp \gamma \Rightarrow \langle \beta, \gamma \rangle$ (converse is false

Theorem (D-Weyman-Schofield):

If $\beta \perp \gamma$ then

$$0 < |\operatorname{Gr}(\beta, V)| = \beta \circ \gamma \text{ for } V \in \operatorname{Rep}_\alpha(Q) \text{ general}$$

$$\begin{array}{c} \| \\ \left(\dim SI(Q,\gamma)_{\langle \beta, \cdot \rangle} \right) \end{array}$$

Suppose $W_1, W_2, \ldots, W_d \in V$ distinct β-dim subreps.

where $d = \beta \circ \gamma$

Then $c^{W_1}, c^{W_2}, \ldots, c^{W_d}$ basis of $SI(Q,\gamma)_{\langle \beta, - \rangle}$.

§10 Exceptional Sequences

Q acyclic quiver, $|Q_0| = n$

V, W representations of Q

def. $V \perp W$ if

$\mathrm{Hom}_Q(V) = \mathrm{Ext}_Q(V,W) = 0$

$(\Leftrightarrow \quad c(V,W) = c^V(W) = c_W(V) = 0)$

$V^\perp = \{ W \in \mathrm{Rep}(Q) \mid V \perp W \}$, $\quad {}^\perp V$

if V indecomposable, not projective then

$V^\perp = {}^\perp(\tau V)$ where τ is AR-transform

if V is sincere (i.e., $V(x) \neq 0$ for all $x \in Q_0$)

then $V^\perp$ and ${}^\perp V$ are equivalent:

$$\begin{tikzcd} {}^\perp V \arrow[r, bend left, "\tau"] & V^\perp \arrow[l, bend left, "\tau^{-1}"] \end{tikzcd}$$

indecomposable representation E called <u>exceptional</u> if $\mathrm{Ext}_Q(E,E)=0$.

if E exceptional then $\varepsilon = \underline{\dim} E$ is real schur root and E has dense orbit in $\mathrm{Rep}_\varepsilon(Q)$.

bijection: real Schur roots $\Longleftrightarrow$ isom. classes of exceptional repr.

Theorem (Schofield):
if $E \in \mathrm{Rep}(Q)$ is exceptional, then $E^\perp$ naturally equivalent to $\mathrm{Rep}(Q(\varepsilon))$ for some acyclic quiver $Q(\varepsilon)$ with $n-1$ elements

def: sequence $(E_1, E_2, \dots, E_m)$ is partial exceptional sequence if:
1. $E_1, E_2, \dots, E_m$ are exceptional
2. $E_i \perp E_j$ for $i<j$.

$m \leq n$ because $(\langle \mathcal{E}_i, \mathcal{E}_j \rangle)_{1 \leq i,j \leq m}$ is lower triangular with 1's on diagonal and rank $\leq n$.

complete exceptional sequence if $m = n$.

Example: Suppose $Q_0 = \{1, 2, \dots, n\}$ and $ha < ta$ for all $a \in Q_1$. Then $S_1, S_2, \dots, S_n$ is exceptional

if $E_1, E_2, \dots, E_m$ exceptional, then $E_1^{\perp} \cap E_2^{\perp} \cap \cdots \cap E_m^{\perp}$ equivalent to $\mathrm{Rep}(Q')$ where Q' acyclic quiver with $n-m$ vertices (Schofield & induction)
Label Q_0' as above and let $S_1', S_2', \dots, S_{n-m}'$ be simples in $\mathrm{Rep}(Q_0')$. Then $E_1, E_2, \to E_m, S_1', S_2', \dots, S_{n-m}'$ is complete exceptional sequence.

Suppose (E_1, E_2) partial exceptional (reduce to $|Q_0|=2$)

$\varepsilon_i = \underline{\dim}\, E_i$. $\langle \varepsilon_1, \varepsilon_1 \rangle = \langle \varepsilon_2, \varepsilon_2 \rangle = 1$, $\langle \varepsilon_1, \varepsilon_2 \rangle = 0$

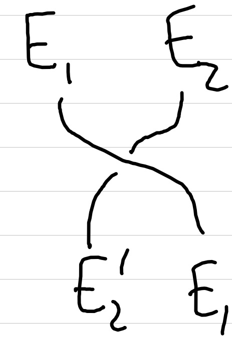
Let $k = \langle \varepsilon_2, \varepsilon_1 \rangle$.

suppose $k \leq 0$. Then $\mathrm{Hom}_Q(E_2, E_1) = 0$ and $\dim \mathrm{Ext}_Q(E_2, E_1) = -k$

Exact sequence: $0 \to E_1^{(-k)} \to E_2' \longrightarrow E_2 \longrightarrow 0$

$$\begin{array}{ccc} -k\varepsilon_1 & \varepsilon_2 - k\varepsilon_1 & \varepsilon_2 \end{array} \qquad \begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix}$$

(E_2', E_1) exceptional

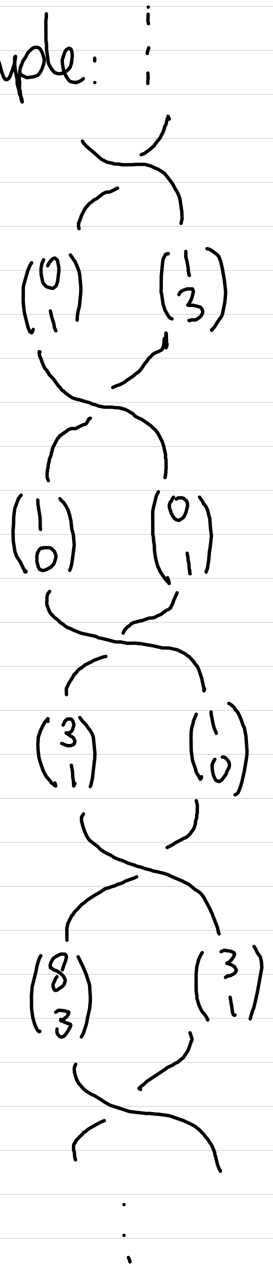


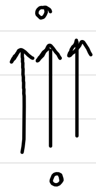
suppose $k \geq 0$ then $\mathrm{Ext}_Q(E_2, E_1) = 0$ and $\mathrm{Hom}_Q(E_2, E_1) = k$

Exact sequence

$$0 \longrightarrow E_2 \longrightarrow E_1^k \longrightarrow \underset{k\varepsilon_1 - \varepsilon_2}{E_2'} \longrightarrow 0 \quad \text{or} \quad 0 \longrightarrow \underset{\varepsilon_2 - k\varepsilon_1}{E_2'} \longrightarrow E_2 \longrightarrow E_1^k \longrightarrow 0$$

Example:

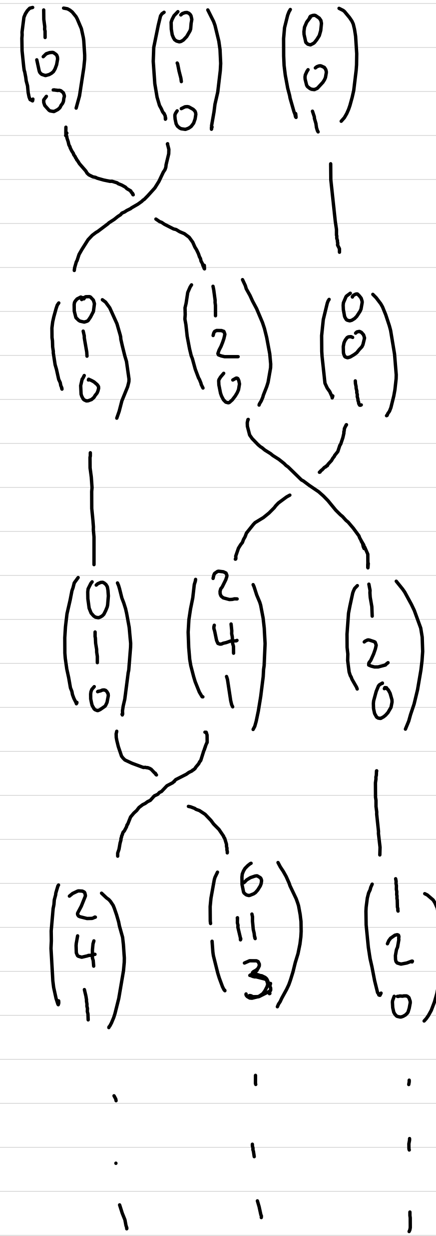




If $|Q_0| = 2$ then we get all complete exceptional sequences by twists.

Theorem (Crawley-Boevey)
There is a transitive group action of the Braid group B_n on all complete exceptional sequences.

Example:



§ 11 The σ-stable decomposition. (D-Weyman)

$\sigma \in \mathbb{Z}^{Q_0}$ weight

if $V \in \mathrm{Rep}_\alpha(Q)$ is σ-semi-stable, it has a Jordan-Hölder filtration

$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{s-1} \subset V_s = V$ with

V_i/V_{i-1} σ-stable for all i.

JH-filtration is not unique, but the quotients $\{ V_i/V_{i-1} \mid 1 \leq i \leq s\}$ are.

Def. Suppose α is σ-semi-stable (i.e., $\mathrm{Rep}_\alpha(Q)^{ss}_\sigma \neq \emptyset$)

we say $\alpha = \alpha_1 \dotplus \alpha_2 \dotplus \cdots \dotplus \alpha_s$ is σ-stable decomposition if a general $V \in \mathrm{Rep}_\alpha(Q)$ has a JH-filtration such that the dimensions of the factors are $\alpha_1, \alpha_2, \ldots, \alpha_s$ in some order.

α is σ-stable
(i.e. $\mathrm{Rep}_\alpha(Q)^s \neq \phi$) $\Leftrightarrow$ α Schur root
for some σ

Schofield: for $\Leftarrow$ one can take $\sigma = \langle \alpha, \cdot \rangle - \langle \cdot, \alpha \rangle$

if $\alpha, \beta \in \mathbb{N}^{Q_0}$ then
$\alpha \perp \beta$ means $\hom_Q(\alpha, \beta) = \mathrm{ext}_Q(\alpha, \beta) = 0$ $\Rightarrow$ $\alpha \circ \beta > 0$
write $c \cdot \alpha$ for $\underbrace{\alpha \dot{+} \alpha \dot{+} \cdots \dot{+} \alpha}_{c}$

Prop. if $\alpha = c_1 \cdot \alpha_1 \dot{+} c_2 \cdot \alpha_2 \dot{+} \cdots \dot{+} c_s \cdot \alpha_s$
is σ-stable decomposition with $\alpha_1, \alpha_2, \ldots, \alpha_s$ distinct
and $c_i \geq 1$ for all i, then
① α_i Schur root $\forall i$
② $\hom(\alpha_i, \alpha_j) = 0$ $\forall i \neq j$
③ after rearranging, $\alpha_i \circ \alpha_j = 1$ for all $i < j$.

generalization of exceptional sequences allowing for imaginary Schur roots:

def.: A sequence $\alpha_1, \alpha_2 - , \alpha_s$ is a <u>Schur sequence</u> if

① $\alpha_1, \alpha_2, \cdots, \alpha_s$ Schur roots

② $\alpha_i \circ \alpha_j = 1$ for $i<j$.

if $\alpha = \alpha_1^{\oplus c_1} \oplus \alpha_2^{\oplus c_2} \oplus \cdots \oplus \alpha_s^{\oplus c_s}$ is canonical decomp.
then after rearranging, $\alpha_1, \alpha_2, -, \alpha_s$ Schur sequence
and $\langle \alpha_j, \alpha_i \rangle \geq 0$ for $i<j$.
and $c_i = 1$ whenever $\langle \alpha_i, \alpha_i \rangle < 0$.
converse is also true

if $\alpha = c_1 \cdot \alpha_1 \dot{+} c_2 \cdot \alpha_2 \dot{+} \cdots \dot{+} c_s \cdot \alpha_s$ is σ-stable decomposition, then, after rearranging, $\alpha_1, \alpha_2, \cdots, \alpha_s$ is Schur sequence and $\langle \alpha_j, \alpha_i \rangle \leq 0$ for $i<j$. moreover $c_i = 1$ whenever $\langle \alpha_i, \alpha_i \rangle < 0$. converse also true.

§12 Variation of weights

$\alpha, \beta \in \mathbb{N}^{Q_0}$, $\sigma = \langle \alpha, \cdot \rangle$

Schofield: $\operatorname{ext}_Q(\alpha,\beta) = \max\limits_{\alpha' \hookrightarrow \alpha} \{ -\langle \alpha', \beta \rangle \} = \max\limits_{\beta \twoheadrightarrow \beta''} \{ -\langle \alpha, \beta' \rangle \}$

(recall $\alpha' \hookrightarrow \alpha \Leftrightarrow \operatorname{ext}_Q(\alpha', \alpha-\alpha') = 0$, gives recursive algorithm for computing $\operatorname{ext}_Q(\alpha,\beta)$)

$$\Sigma(Q,\beta) = \{ \sigma \in \mathbb{Z}^{Q_0} \mid SI(Q,\beta)_\sigma \neq 0 \}$$

assume $\sigma(\beta) = 0$, then:

$\sigma \in \Sigma(Q,\beta) \Leftrightarrow \operatorname{ext}_Q(\alpha,\beta) = 0 \Leftrightarrow \forall \beta \twoheadrightarrow \beta' \ \sigma(\beta') \geq 0 \Leftrightarrow \forall \beta' \hookrightarrow \beta \ \sigma(\beta') \leq 0$

So $\Sigma(Q,\beta)$ is saturated: $\Sigma(Q,\beta) = \mathbb{R}_+ \Sigma(Q,\beta) \cap \mathbb{Z}^{Q_0}$.

Bijection:

r-dimensional faces of $\mathbb{R}_+\Sigma(Q,\beta)$ $\longleftrightarrow$ sets $\{\gamma_1,\gamma_2,\dots,\gamma_r\}$ such that $\gamma_1,\gamma_2,\dots,\gamma_r$ is a Schur sequence with $\langle\gamma_j,\gamma_i\rangle\leq 0$ for all $i<j$, and there exist $b_1,b_2,\dots,b_r\geq 1$ such that $\beta=\sum b_i\gamma_i$ and $b_i=1$ whenever $\langle\gamma_i,\gamma_i\rangle<0$.

Example:

$$Q=\begin{tikzcd} & \circ \arrow[d] & \\ \circ \arrow[r] & \circ & \circ \arrow[l] \\ & \circ \arrow[u] & \end{tikzcd}$$

$$\beta=\begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \\ & 1 & \end{smallmatrix}$$

$\mathbb{R}_+\Sigma(Q,\beta)$ has dim 4.

dim 3 faces "walls"

$$\boxed{\begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \\ & 1 & \end{smallmatrix}}=\boxed{\begin{smallmatrix} & 0 & \\ 1 & 2 & 1 \\ & 1 & \end{smallmatrix}}\dotplus\boxed{\begin{smallmatrix} & 1 & \\ 0 & 0 & 0 \\ & 0 & \end{smallmatrix}}$$ 4 by symmetry

$$\boxed{\begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \\ & 1 & \end{smallmatrix}}=\boxed{\begin{smallmatrix} & 1 & \\ 0 & 1 & 0 \\ & 0 & \end{smallmatrix}}\dotplus\boxed{\begin{smallmatrix} & 0 & \\ 1 & 1 & 1 \\ & 1 & \end{smallmatrix}}$$ 4 by symm.

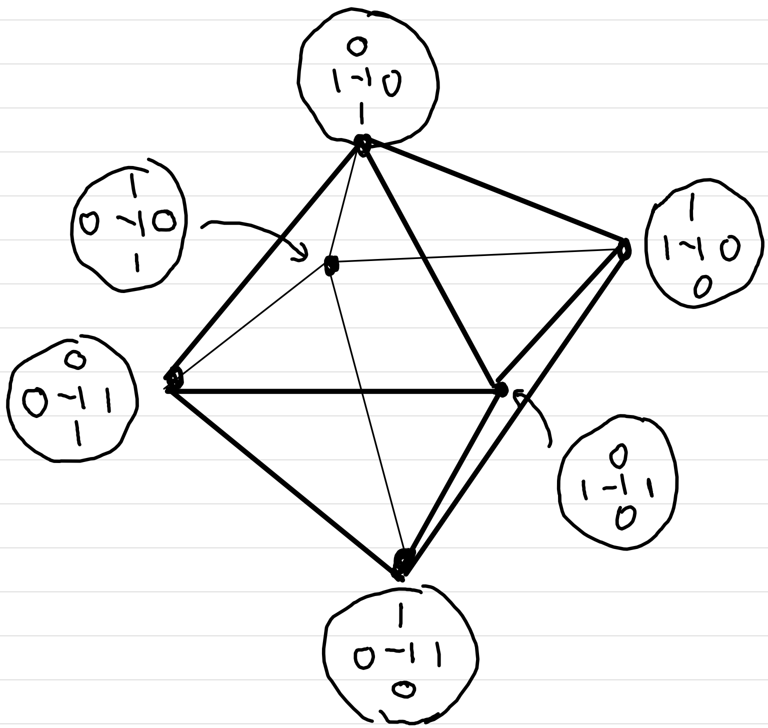
dim 2 faces

$$\boxed{\begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \\ & 1 & \end{smallmatrix}}=\boxed{\begin{smallmatrix} & 1 & \\ 0 & 1 & 0 \\ & 0 & \end{smallmatrix}}\dotplus\boxed{\begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \\ & 1 & \end{smallmatrix}}\dotplus\boxed{\begin{smallmatrix} & 0 & \\ 0 & 0 & 1 \\ & 0 & \end{smallmatrix}}$$ 12 by symmetry

dim 1 faces

$$\boxed{\begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \\ & 1 & \end{smallmatrix}}=\boxed{\begin{smallmatrix} & 1 & \\ 0 & 1 & 0 \\ & 0 & \end{smallmatrix}}\dotplus\boxed{\begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \\ & 0 & \end{smallmatrix}}+\boxed{\begin{smallmatrix} & 0 & \\ 0 & 0 & 0 \\ & 1 & \end{smallmatrix}}\dotplus\boxed{\begin{smallmatrix} & 0 & \\ 0 & 0 & 1 \\ & 0 & \end{smallmatrix}}$$ 6 by symmetry

$\Sigma(Q,\alpha)$ is cone over



§1.3 Representation Theory of GL_n

irreducible representations:

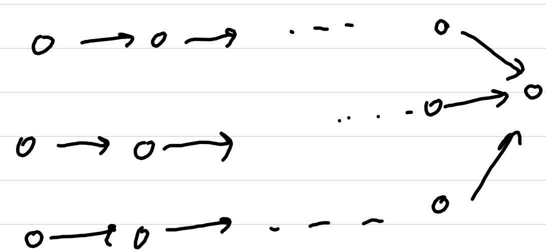
V_λ, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

$c_{\lambda,\mu}^{\nu} = \operatorname{mult}(V_\mu \subset V_\lambda \otimes V_\mu) = \dim (V_\lambda \otimes V_\mu \otimes V_\nu^*)^{GL_n}$

Littlewood Richardson coeff.

$LR_n = \{(\lambda, \mu, \nu) \in (\mathbb{Z}^n)^3 \mid c_{\lambda,\mu}^{\nu} \neq 0\}$

Let $Q =$



$$\beta = \begin{array}{c} 1\ 2 \cdots n-1 \\ 1\ 2 \cdots n-1\ \ n \\ 1\ 2 \cdots n-1 \end{array}$$

then $\Sigma(Q,\beta) \times \mathbb{Z}^2 \overset{\sim}{=} LR_n$

more generally all numbers $\gamma \circ \delta$ can be expressed as LR-coeffs., $\gamma, \delta \in \mathbb{N}^{Q_0}$

Saturation of $\Sigma(Q,\beta)$ implies Klyachko's Saturation conjecture (proved by Knutson-Tao first)

D-Weyman: $SI(Q,\beta)_{n\sigma}$ is polynomial in n. (for any Q,β)

so $c^{n\nu}_{n\lambda,n\mu}$ is polynomial in n

(conjecture: nonneg. coeffs)

Buch?

Fulton conjecture, proved by Knutson-Tao-Woodward, states $c^{\nu}_{\lambda,\mu}=1 \Rightarrow c^{n\nu}_{n\lambda,n\mu}=1$ for all $n\geq 0$.

generalizes to:

Belkale: if $\dim SI(Q,\beta)_{\sigma}=1$ then $\dim SI(Q,\beta)_{n\sigma}=1$ for all n.

$\Rightarrow$ if $\alpha\circ\beta=1$ then $p\alpha\circ q\beta=1$ for all $p,q\geq 0$.

§14 Semi-invariants for algebras.

recall: $V \in \mathrm{Rep}_\alpha(Q)$, $W \in \mathrm{Rep}_\beta(Q)$, $\langle \alpha, \beta \rangle = 0$

(*) $0 \to P_1 \longrightarrow P_0 \to V \to 0$ canonical projective resolution

$$0 \to \mathrm{Hom}_Q(V,W) \longrightarrow \mathrm{Hom}_Q(P_0,V) \xrightarrow{d^V_W} \mathrm{Hom}_Q(P_1,V) \longrightarrow \mathrm{Ext}_Q(V,W) \to 0$$

$$c(V,W) = \det d^V_W = c^V(W) = c_W(V)$$

$$c^V \in SI(Q,\beta)_{\langle \alpha, \cdot \rangle}, \quad c_W \in SI(Q,\alpha)_{-\langle \cdot, \beta \rangle}$$

instead of canonical proj. resolution we can also take minimal proj. res.
still gives same c^V, c_W (up to constant scalar)

Thm (D-Weyman): $SI(Q,\beta)$ generated (in fact spanned) by Schofield invariants c^V where $\alpha \in \mathbb{N}^{Q_0}$, $\langle \alpha, \beta \rangle = 0$ and $V \in \mathrm{Rep}_\alpha(Q)$

Let $\sigma = -\langle \cdot, \beta \rangle$. Then we only need V's that are σ-stable, otherwise there is exact sequence $0 \to \underset{\sigma}{\underset{\#}{V_1}} \to V \to \underset{\sigma}{\underset{\#}{V_2}} \to 0$ with $c^V = c^{V_1} c^{V_2}$.

in particular, we only need to consider α for which α is σ-stable (and therefore a Schur root.)

$I \subseteq KQ$ admissible ideal

$A = KQ/I$ basic algebra, $\beta \in \mathbb{N}^{Q_0}$

$\mathrm{Rep}_\beta(A) = \{W \in \mathrm{Rep}_\beta(Q) \mid W(p) = 0 \ \forall p \in I\} \underset{\text{closed}}{\subseteq} \mathrm{Rep}_\beta(Q)$

GL_β-invariant

We can again construct GIT quotient following A. King.

$$\mathrm{Rep}_\beta(A) = \bigcup_{i=1}^{s} \mathrm{Rep}_\beta(A)^{[i]} \quad \text{irreducible components}$$

$$SI(Q,\beta) = K[\mathrm{Rep}_\beta(Q)]^{SL_\beta} = \bigoplus_{\sigma \in \mathbb{Z}^{Q_0}} SI(Q,\beta)_\sigma$$

$$SI(A,\beta)^{[i]} = K[\mathrm{Rep}_\beta(A)^{[i]}]^{SL_\beta} = \bigoplus_{\sigma} SI(A,\beta)_\sigma^{[i]}$$

We assume char $K = 0$

$SI(Q,\beta) \twoheadrightarrow SI(A,\beta)^{[i]}$ onto for all i.

if $V \in \operatorname{Rep}_\alpha(A)$, $\quad W \in \operatorname{Rep}_\beta(A)$ then

Suppose $P_1 \to P_0 \to V \to 0$ is minimal presentation in $\operatorname{Rep}(A)$.

For $W \in \operatorname{Rep}_\beta(A)$

$$0 \to \operatorname{Hom}_Q(V,W) \to \operatorname{Hom}(P_0,W) \xrightarrow{d_W^V} \operatorname{Hom}(P_1,W) \to E_Q(V,W) \to 0$$

if d_W^V is square matrix then define $c^V(W) = \det(d_W^V)$.

Now $c^V \in SI(A,\beta)^{[i]}$

Theorem (D-Weyman): $SI(A,\beta)^{[i]}$ is spanned by c^V's

def: $\operatorname{Rep}_\beta(A)^{[i]}$ called faithful component if for all $a \in A$,

$$\left[\forall W \in \operatorname{Rep}_\beta(A)^{[i]} \quad aW = 0\right] \Rightarrow a = 0.$$

Thm: if $\mathrm{Rep}_\beta(A)^{[i]}$ is faithful, then $SI(A,\beta)^{[i]}$ spanned by c^V, $V \in \mathrm{Rep}(A)$ and V has proj. dim ≤ 1.

(if not faithful, replace A by A/J so that $\mathrm{Rep}_\beta(A)^{[i]} = \mathrm{Rep}_\beta(A/J)^{[i]}$ is faithful for A/J)

suppose

$$P_1 \xrightarrow{\phi} P_0 \to V \to 0$$

minimal presentation.

$P_0 = \bigoplus_{x \in Q_0} P_x^{h(x)}$, $P_1 \to \bigoplus_{x \in Q_0} P_x^{\bar{h}(x)}$, $g = h - \bar{h}$ is g-vector of ϕ

then c^V has weight g

We only have to consider c^V's where ϕ general.

if $\phi : P_1 \to P_0$ is general then $h(x)\bar{h}(x) = 0 \ \forall x \in Q_0$.

$h = g_+ := \max\{g, 0\}$, $\bar{h} = g_- = \max\{0, -g\}$.

Let $g \in \mathbb{Z}^{Q_0}$, $h = g_+$, $\bar{h} = g_-$, $P_0 = \bigoplus_{x \in Q_0} P_x^{h(x)}$, $P_1 = \bigoplus_{x \in Q_0} P_x^{\bar{h}(x)}$.

group $G = \mathrm{Aut}(P_0) \times \mathrm{Aut}(P_1)$ acts on $\mathrm{Hom}(P_1, P_0)$

(G not be reductive)

More generally one can define $E_Q(\phi, \phi')$
if $\phi : P_1 \longrightarrow P_0$ ϕ' $P_1' \to P_0'$ with g-vectors g and g'
and $e_Q(g, g')$ generic value of $\dim E_Q(\phi, \phi')$

Derksen-Fei: One can define a canonical decomposition
of g-vectors, $g = g_1 \oplus g_2 \oplus \cdots \oplus g_s$ is canonical decomposition
if $g_1, g_2, \dots, g_s$ are undecomposable and $e(g_i, g_j) = 0$ for $i \neq j$.