Tilting in functor categories

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The category $\mathcal{P}_d$ of strict polynomial functors of degree d

$\mathbf{k}$ is a field of caharacteristic $p > 0$, $\Gamma^d(V) := (V^{\otimes d})^{\Sigma_d}$

An object of $\mathcal{P}_d$ is determined by:

1. $V \mapsto F(V)$,
2. $F_{V,W} : \Gamma^d(\mathrm{Hom}_{\mathbf{k}}(V, W)) \longrightarrow \mathrm{Hom}_{\mathbf{k}}(F(V), F(W))$
   satisfying the compatibility conditions.

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Evaluation $F \mapsto F(\mathbf{k}^n)$ endows $\mathbf{k}^n$ with a structure of representation of $GL_n(\mathbf{k})$.

When $n \geq d$ it yields an equivalence of abelian categories

$$\mathcal{P}_d \simeq \Gamma^d(\mathrm{End}_{\mathbf{k}}(\mathbf{k}^n))\text{-mod} =: S_{n,d}(\mathbf{k})\text{-mod}$$

Examples of polynomial functors, parameters

$$
\begin{aligned}
&V \rightsquigarrow V^{\otimes d} && (I^d),\\
&V \rightsquigarrow (V^{\otimes d})_{\Sigma_d} && (S^d),\\
&V \rightsquigarrow (V^{\otimes d})^{\Sigma_d} && (\Gamma^d),\\
&V \rightsquigarrow ((V^{\otimes d})^{alt})^{\Sigma_d} \simeq ((V^{\otimes d})^{alt})_{\Sigma_d} && (\Lambda^d),
\end{aligned}
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If char($\mathbf{k}$)=p, $p > 0$,

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\begin{aligned}
&V \rightsquigarrow V^{(1)} && (I^{(1)}),\\
&F^{(1)} := F \circ I^{(1)}.
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Functors with parameters: $U \in \mathbf{k}\mathrm{-mod}^f$, $F_U(V) := F(U \otimes V)$.
We have: $\mathrm{Hom}_{\mathcal{P}_d}(\Gamma^d_{U^*}, F) \simeq F(U)$, (Yoneda lemma),
hence if $\dim(U) \geq d$, then $\Gamma^d_{U^*}$ is a projective generator $\mathcal{P}_d$.

Schur, Weyl and simple objects, Kuhn duality

Young diagram of weight d: $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\sum \lambda_j = d$.

$$S_\lambda := \mathrm{im}(\Lambda^{\lambda_1} \otimes \ldots \otimes \Lambda^{\lambda_k} \longrightarrow I^d \longrightarrow S^{\widetilde{\lambda}_1} \otimes \ldots \otimes S^{\widetilde{\lambda}_s}),$$
$$W_\lambda := \mathrm{im}(\Gamma^{\widetilde{\lambda}_1} \otimes \ldots \otimes \Gamma^{\widetilde{\lambda}_s} \longrightarrow I^d \longrightarrow \Lambda^{\lambda_1} \otimes \ldots \otimes \Lambda^{\lambda_k}),$$

The complete set (of classes of isomorphism) of simples in $\mathcal{P}_d$:
$$F_\lambda := \mathrm{im}(W_\lambda \longrightarrow \Lambda^{\lambda_1} \otimes \ldots \otimes \Lambda^{\lambda_k} \longrightarrow S_\lambda)$$

$F_\lambda \hookrightarrow S_\lambda$, $W_\lambda \twoheadrightarrow F_\lambda$,... ($\mathcal{P}_d$ is highest weight actegory)

$$F^{\#}(V) := (F(V^*)^*,$$
$$(S^d)^{\#} = \Gamma^d,\ (\Lambda^d)^{\#} = \Lambda^d,\ (S_\lambda)^{\#} = W_\lambda,\ (F_\lambda)^{\#} = F_\lambda.$$

Tilting in $\mathcal{P}_d$ aka Koszul duality aka Ringel duality

If $\dim(U) \geq d$, then $\Lambda^d_{U^*}$ is a tilting object in $\mathcal{P}_d$, hence we have:

$$\mathcal{D}(\mathcal{P}_d) \simeq \mathcal{D}(\mathrm{End}_{\mathcal{P}_d}(\Lambda^d_{U^*})^{op}\text{--mod}) \simeq \mathcal{D}(\Gamma^d(\mathrm{End}_{\mathbf{k}}(U))\text{--mod}) \simeq \mathcal{D}\mathcal{P}_d,$$

or we can directly define an auto-equivalence of $\mathcal{D}\mathcal{P}_d$ given as:

$$\Theta(F^\bullet)(V) := \mathrm{RHom}_{\mathcal{P}_d}(\Lambda^d_{V^*}, F^\bullet)$$

One can compare this with the Yoneda lemma:

$$\mathrm{Hom}_{\mathcal{P}_d}(\Gamma^d_{V^*}, F) = F(V).$$

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Θ enjoys nice properties:

$$\Theta(S^d) = \Lambda^d$$
$$\Theta(S_\lambda) = W_{\tilde{\lambda}}$$
$$\Theta(I^{(1)}) = I^{(1)}[-(p-1)]$$

Abelian vs. triangulated case

Theorem (Gabriel) Let $\mathcal{A}$ be an AB5 category and let $T \in \mathcal{A}$ satisfies the conditions:

- T generates $\mathcal{A}$ (ie. if $X \neq 0$ then $\mathrm{Hom}_{\mathcal{A}}(T, X) \neq 0$).
- T is projective.
- T is compact (ie. $\mathrm{Hom}_{\mathcal{A}}(T, -)$ commutes with infinite sums).

Then the functor: $X \mapsto \mathrm{Hom}_{\mathcal{A}}(T, X)$ yields an equivalence of abelian categories:

$$\mathcal{A} \simeq (\mathrm{End}_{\mathcal{A}}(T)^{op}\text{-mod}).$$

Theorem (Beilinson, Keller,...) Let $\mathcal{A}$ be an AB5 category and let $T^\bullet \in \mathrm{Kom}(\mathcal{A})$ satisfies the conditions:

- $T^\bullet$ generates $\mathcal{D}(\mathcal{A})$.
- $T^\bullet$ is compact.

Then the functor: $X^\bullet \mapsto \mathrm{RHom}_{\mathcal{A}}(T^\bullet, X^\bullet)$ yields an equivalence of triangulated categories:

$$\mathcal{D}(\mathcal{A}) \simeq \mathcal{D}(\mathrm{REnd}_{\mathcal{A}}(T^\bullet)^{op}\text{-dgmod}).$$

Collapsing conjecture and formality

Let $\mathcal{D}(\mathcal{P}_d^{(1)})$ be the full subcategory of $\mathcal{D}(\mathcal{P}_{pd})$ spanned by $F^{(1)}$ for $F \in \mathcal{P}_d$.
$\mathcal{D}(\mathcal{P}_d^{(1)})$ is coreflective (ie. inclusion admits the right adjoint) and generated by $\Gamma_{U^*}^{d(1)}$ when $\dim(U) \geq d$. Therefore:

$$\mathcal{D}(\mathcal{P}_d^{(1)}) \simeq \mathcal{D}(\mathrm{REnd}_{\mathcal{P}_{pd}}(\Gamma_{U^*}^{d(1)})^{op}\text{-dgmod}).$$

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Theorem/"Collapsing conjecture" (MC)
There is a quasi-isomorphism of dg-algebras:

$$\mathrm{REnd}(\Gamma^{d(1)}_{U^*}) \simeq H^*(\mathrm{End}(\Gamma^{d(1)}_{U^*})) = \mathrm{Ext}^*(\Gamma^{d(1)}_{U^*}, \Gamma^{d(1)}_{U^*}) = \Gamma^d(\mathrm{End}(U) \otimes A),$$

where $A := \mathrm{Ext}^*_{\mathcal{P}_p}(I^{(1)}, I^{(1)}) \simeq \mathbf{k}[x]/x^p$, for $\deg(x) = 2$.
Hence there is an equivalence of triangulated categories:

$$\mathcal{D}(\mathcal{P}_d^{(1)}) \simeq \mathcal{D}(\Gamma^d(\mathrm{End}_{\mathbf{k}}(U) \otimes A) - \mathrm{dgmod}).$$

Affine strict polynomial functors

An object of $\mathcal{P}_d^{af}$ is determined by:

1. For a fg. free graded A-module V, the graded $\mathbf{k}$-module $F(V)$
2. For any pair V, W of fg. free graded A-modules, the graded $\mathbf{k}$-linear map: $F_{V,W} : \Gamma^d(\mathrm{Hom}_A(V, W)) \longrightarrow \mathrm{Hom}_{\mathbf{k}}(F(V), F(W))$ satisfying the compatibility conditions.

$\mathrm{Hom}_{\mathcal{P}_d^{af}}(F, G) := \mathrm{Nat}^{gr}(F, G)$

There is an equivalence of triangulated categories:

$$\mathcal{D}(\mathcal{P}_d^{(1)}) \simeq \mathcal{D}(\mathcal{P}_d^{af})$$

Towards $\mathrm{Ext}^*_{\mathcal{P}_d}(S_\lambda, S_\mu)$

How to compute $\mathrm{Ext}^*_{\mathcal{P}_{pd}}(S^{pd}, \Lambda^{pd})$ (for $p|d$)? (done by Akin)

Consider the de Rham complex $\mathbf{S}^{pd}$:

$$0 \to S^{pd} \to \ldots \to S^{pd-i} \otimes \Lambda^i \to S^{pd-i-1} \otimes \Lambda^{i+1} \to \ldots \to \Lambda^{pd} \to 0.$$

Theorem (Cartier) $H^*(\mathbf{S}^{pd}) = \mathbf{S}^{d(1)}$.

Hence one can proceed by induction on d as follows:

- Compute $\mathrm{Ext}^*_{\mathcal{P}_{pd}}(H^*(\mathbf{S}^{pd}), \Lambda^{pd}))$.
- Compute $\mathrm{HExt}^*_{\mathcal{P}_{pd}}(\mathbf{S}^{pd}, \Lambda^{pd}))$ by using $E_2^{**} = \mathrm{Ext}^*_{\mathcal{P}_{pd}}(H^*(\mathbf{S}^{pd}), \Lambda^{pd})) \Rightarrow \mathrm{HExt}^*_{\mathcal{P}_{pd}}(\mathbf{S}^{pd}, \Lambda^{pd}))$.
- Compute $\mathrm{Ext}^*_{\mathcal{P}_{pd}}(S^{pd}, \Lambda^{pd})$ by using $E_1^{**} = \mathrm{Ext}^*_{\mathcal{P}_{pd}}(\mathbf{S}^{pd}, \Lambda^{pd})) \Rightarrow \mathrm{HExt}^*(\mathbf{S}^{pd}, \Lambda^{pd}))$.

Schur-de Rham complex (MC, inspired by [ABW])

$$S^d = (I^{\otimes d})_{\Sigma_d} \qquad \mathbf{S}^d = ((I \xrightarrow{\mathrm{id}} I)^{\otimes d})_{\Sigma_d}$$

Then for any Young diagram of weight d:

$$S_\lambda = s_\lambda(I^{\otimes d}) \qquad \mathbf{S}_\lambda = s_\lambda((I \xrightarrow{\mathrm{id}} I)^{\otimes d})$$

We have:

$$0 \longrightarrow S_\lambda \longrightarrow \ldots \longrightarrow W_{\widetilde{\lambda}} \longrightarrow 0$$

Problem: Compute $H^*(\mathbf{S}_\lambda)$.

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Problem: Compute $H^*(\mathbf{S}_\lambda)$.

Alternatively, we can describe $\mathbf{S}_\lambda$ as:

$$\mathbf{S}_\lambda(V) := \mathrm{Hom}_{\mathcal{P}_d}(\mathbf{S}^d_{V^*}, S_\lambda)^\#$$

One can study the functor:

$$\mathcal{R}(F^\bullet)(V) := \mathrm{RHom}_{\mathcal{P}_d}(\mathbf{S}^d_{V^*}, F^\bullet)^\#$$